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Von Neumann Regular and Related Elements in Commutative Rings

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Abstract. Let R be a commutative ring with nonzero identity. In this paper, we study the von Neumann regular elements of R. We also study the idempotent elements, π-regular elements, the von Neumann local elements, and the clean elements of R. Finally, we investigate the subgraphs of the zero-divisor graph $\Gamma(R)$ of R induced by the above elements.

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1 Introduction

Let R be a commutative ring with nonzero identity. Recall that R is *von Neumann regular* if for every $a \in R$, there is $x \in R$ such that $a^2x = a$, that R is *π-regular* if for every $a \in R$, there are $x \in R$ and an integer $n \geq 1$ such that $a^{2n}x = a^n$, and that R is *Boolean* if $a^2 = a$ for every $a \in R$. Thus a Boolean ring is von Neumann regular and a von Neumann regular ring is π-regular. Moreover, R is π-regular (resp., von Neumann regular) if and only if R is zero-dimensional (resp., reduced and zero-dimensional) [19, Theorem 3.1, page 10]; so R is π-regular if and only if $R/\mathrm{nil}(R)$ is von Neumann regular. Specializing to elements, we define $a \in R$ to be a

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von Neumann regular element of R if $a^2x = a$ for some $x \in R$. Similarly, we define $a \in R$ to be a *π-regular element* of R if $a^{2n}x = a^n$ for some $x \in R$ and $n \geq 1$. Let $\mathrm{Idem}(R) = \{a \in R \,|\, a^2 = a\}$, $\mathrm{vnr}(R) = \{a \in R \,|\, a \text{ is von Neumann regular}\}$, and $\pi\text{-}r(R) = \{a \in R \,|\, a \text{ is } \pi\text{-regular}\}$. Thus $\mathrm{Idem}(R) \subseteq \mathrm{vnr}(R) \subseteq \pi\text{-}r(R)$ and R is a Boolean (resp., von Neumann regular, π-regular) ring if and only if $\mathrm{Idem}(R) = R$ (resp., $\mathrm{vnr}(R) = R$, $\pi\text{-}r(R) = R$).

Following [11], we define R to be a *von Neumann local ring* if either $a \in \mathrm{vnr}(R)$ or $1 - a \in \mathrm{vnr}(R)$ for every $a \in R$. As in [27], we say that R is a *clean ring* if every element of R is the sum of a unit and an idempotent of R. Specializing to elements again, we define $a \in R$ to be a *von Neumann local element* of R if either $a \in \mathrm{vnr}(R)$ or $1 - a \in \mathrm{vnr}(R)$, and we define $a \in R$ to be a *clean element* of R if a is the sum of a unit and an idempotent of R. Let $\mathrm{vnl}(R) = \{a \in R \,|\, a \text{ is von Neumann local}\}$ and $\mathrm{cln}(R) = \{a \in R \,|\, a \text{ is clean}\}$. Thus R is a von Neumann local (resp., clean) ring if and only if $\mathrm{vnl}(R) = R$ (resp., $\mathrm{cln}(R) = R$).

We have $\mathrm{Idem}(R) \subseteq \mathrm{vnr}(R) \subseteq \pi\text{-}r(R) \subseteq \mathrm{cln}(R)$ and $\mathrm{vnr}(R) \subseteq \mathrm{vnl}(R) \subseteq \mathrm{cln}(R)$ for any commutative ring R. Moreover, all inclusions may be strict and $\pi\text{-}r(R)$ and $\mathrm{vnl}(R)$ need not be comparable (this happens if $R = \mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_4$). However, if R is an integral domain, then R is von Neumann regular if and only if R is π-regular, if and only if R is a field; and R is von Neumann local if and only if R is clean, if and only if R is quasilocal. More generally, $\mathrm{vnr}(R) = \pi\text{-}r(R) = U(R) \cup \{0\}$ and $\mathrm{vnl}(R) = \mathrm{cln}(R) = \{0, 1\} + U(R)$ when R is an integral domain. Thus these notions are more interesting for rings with nonzero zero-divisors.

In Section 2, we collect elementary results about von Neumann regular elements that will be used throughout this paper. Most of these results are well known in the von Neumann regular ring context. We show that every element of R is either von Neumann regular or nilpotent if and only if R is zero-dimensional and either reduced or quasilocal, and that a non-domain R is von Neumann regular (resp., Boolean) if and only if its zero-divisors are all von Neumann regular (resp., idempotent). We also give necessary and sufficient conditions for $\mathrm{vnr}(R)$ to be a subring of R when $2 \in U(R)$. In Section 3, we investigate $\mathrm{vnr}(T)$ for several ring extensions $R \subseteq T$. In particular, we consider $\mathrm{vnr}(R[X])$, $\mathrm{vnr}(R[[X]])$, $\mathrm{vnr}(R(+)M)$, and $\mathrm{vnr}(R \bowtie I)$.

In Section 4, we study π-regular elements. We give several results for π-regular elements analogous to those for von Neumann regular elements. In particular, we show that $\pi\text{-}r(R) = \mathrm{vnr}(R) + \mathrm{nil}(R)$, that $\pi\text{-}r(R) = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$ if and only if either $\mathrm{vnr}(R) = U(R) \cup \{0\}$ or $\mathrm{nil}(R) = \{0\}$, and that a ring R with $\mathrm{nil}(R) \subsetneq Z(R)$ is π-regular if and only if its zero-divisors are all π-regular elements of R. We also investigate $\pi\text{-}r(T)$ for several ring extensions $R \subseteq T$.

In Section 5, we study von Neumann local elements, and in Section 6, we consider clean elements. We give several results for von Neumann local elements and clean elements analogous to those for von Neumann regular elements and π-regular elements. However, unlike $\mathrm{Idem}(R)$, $\mathrm{vnr}(R)$, and $\pi\text{-}r(R)$, the sets $\mathrm{vnl}(R)$ and $\mathrm{cln}(R)$ need not be multiplicatively closed. We also investigate $\mathrm{vnl}(T)$ and $\mathrm{cln}(T)$ for several ring extensions $R \subseteq T$.

In Section 7, we investigate the induced subgraphs $\Gamma(\mathrm{Idem}(R))$, $\Gamma(\mathrm{vnr}(R))$, $\Gamma(\pi\text{-}r(R))$, $\Gamma(\mathrm{vnl}(R))$, and $\Gamma(\mathrm{cln}(R))$ of the zero-divisor graph $\Gamma(R)$ of R determined by the idempotent, von Neumann regular, π-regular, von Neumann local, and clean elements of $Z(R)$, respectively. In particular, we show that $\Gamma(\mathrm{Idem}(R))$, $\Gamma(\mathrm{vnr}(R))$, and $\Gamma(\pi\text{-}r(R))$ are each connected with diameter at most three, that each has girth at most four if it contains a cycle, and that $\Gamma(\mathrm{Idem}(R))$ and $\Gamma(\mathrm{vnr}(R))$ are uniquely complemented. However, $\Gamma(\mathrm{vnl}(R))$ and $\Gamma(\mathrm{cln}(R))$ need not be connected, and $\Gamma(\pi\text{-}r(R))$, $\Gamma(\mathrm{vnl}(R))$, and $\Gamma(\mathrm{cln}(R))$ need not be uniquely complemented.

Throughout, R will be a commutative ring with nonzero identity, $Z(R)$ its set of zero-divisors, $U(R)$ its group of units, $\mathrm{nil}(R)$ its ideal of nilpotent elements, $J(R)$ its Jacobson radical, and $T(R)$ its total quotient ring. For $A \subseteq R$, let $A^* = A\backslash\{0\}$. Recall that R is *reduced* if $\mathrm{nil}(R) = \{0\}$. The Krull dimension of R will be denoted by $\dim(R)$, and the characteristic of R will be denoted by $\mathrm{char}(R)$. For a homomorphism $f : R \to S$ of commutative rings, we assume that $f(1) = 1$. As usual, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{Z}_n$ will denote the integers, rational numbers, and integers modulo n, respectively. For any undefined notation or terminology, see [17], [19] or [25].

2 Von Neumann Regular elements

Theorem 2.1. *Let R and S be commutative rings, and let $\{R_\alpha\}$ be a family of commutative rings.*

(1) *Let $a \in R$. If $a^2x = a$ for $x \in R$, then $ax \in \mathrm{Idem}(R)$.*

(2) [1, Proposition 3.10] $\mathrm{vnr}(R)$ *is multiplicatively closed.*

(3) $\mathrm{vnr}(R) \cap \mathrm{nil}(R) = \{0\}$.

(4) $U(R) \cup \mathrm{Idem}(R) \subseteq \mathrm{vnr}(R) \subseteq U(R) \cup Z(R)$.

(5) $\mathrm{vnr}(R) = U(R) \cup \{0\}$ *if and only if* $\mathrm{Idem}(R) = \{0, 1\}$. *In particular,* $\mathrm{vnr}(R) = U(R) \cup \{0\}$ *if R is either an integral domain or quasilocal.*

(6) $\mathrm{vnr}(R)$ *contains a nonzero nonunit if and only if* $\{0, 1\} \subsetneq \mathrm{Idem}(R)$.

(7) $\mathrm{vnr}\big(\prod R_\alpha\big) = \prod \mathrm{vnr}(R_\alpha)$. *In particular, $\prod R_\alpha$ is von Neumann regular if and only if each R_α is von Neumann regular.*

(8) *Let $f : R \to S$ be a homomorphism of commutative rings. Then $f(\mathrm{vnr}(R)) \subseteq \mathrm{vnr}(S)$. In particular, $\mathrm{vnr}(R) \subseteq \mathrm{vnr}(S)$ when R is a subring of S, and any homomorphic image of a von Neumann regular ring is von Neumann regular.*

It is well known that if R is a von Neumann regular ring, then for every $a \in R$, there is $x \in U(R)$ such that $a^2x = a$. Moreover, $a = ue$ for some $u \in U(R)$ and $e \in \mathrm{Idem}(R)$. We next show that these two results, plus several others, hold in general for elements of $\mathrm{vnr}(R)$ (cf. [19, Corollary 3.3, page 11] and [7, Section 2]).

Theorem 2.2. *Let R be a commutative ring. Then the following statements are equivalent for $a \in R$:*

(1) $a \in \mathrm{vnr}(R)$.

(2) $a^2u = a$ *for some* $u \in U(R)$.

(3) $a = ue$ *for some* $u \in U(R)$ *and* $e \in \mathrm{Idem}(R)$.

(4) $ab = 0$ *for some* $b \in \mathrm{vnr}(R)\backslash\{a\}$ *with* $a + b \in U(R)$.

(5) $ab = 0$ *for some* $b \in R$ *with* $a + b \in U(R)$.

Proof. (1)⇒(2) Suppose that $a^2x = a$ for $x \in R$. Then $e = ax \in \mathrm{Idem}(R)$ by Theorem 2.1(1); so $1 - e \in \mathrm{Idem}(R)$ and $a(1-e) = 0$. Thus $u = ex + 1 - e \in U(R)$ since $u(a + 1 - e) = 1$, and $a^2u = a^2(ex + 1 - e) = a^2ex + a^2(1-e) = a$.

(2)⇒(3) Suppose that $a^2v = a$ for some $v \in U(R)$. Let $e = av \in \mathrm{Idem}(R)$ and $u = v^{-1} \in U(R)$. Then $ue = v^{-1}(av) = a$.

(3)⇒(4) Suppose that $a = ue$ for $u \in U(R)$ and $e \in \mathrm{Idem}(R)$. Let $b = u(1-e)$; so $b \neq a$. Note that $b^2u^{-1} = u^2(1-e)^2u^{-1} = u(1-e) = b$; so $b \in \mathrm{vnr}(R)$. Then $ab = (ue)(u(1-e)) = 0$ and $a + b = ue + u(1-e) = u \in U(R)$.

(4)⇒(5) This is clear.

(5)⇒(1) Suppose that $ab = 0$ and $a + b = u \in U(R)$ for some $b \in R$. Then $au = a(a+b) = a^2 + ab = a^2$. Thus $a^2u^{-1} = (au)u^{-1} = a$; so $a \in \mathrm{vnr}(R)$. □

Let $a \in \mathrm{vnr}(R)$. Then $a^2x = a$ for some $x \in R$. Note that x need not be unique since we may replace x by any $y \in x + \mathrm{ann}(a^2)$.

Theorem 2.3. [28, Lemma 4] *Let R be a commutative ring and $a \in \mathrm{vnr}(R)$. Then there is a unique $x \in R$ with $a^2x = a$ and $x^2a = x$.*

Since $\mathrm{vnr}(R) \cap \mathrm{nil}(R) = \{0\}$, it is natural to ask when $R = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$, i.e., when is every non-nilpotent element of R von Neumann regular? We next show that this happens only in the two extreme cases: either $\mathrm{nil}(R) = \{0\}$ and $\mathrm{vnr}(R) = R$, in which case R is von Neumann regular; or $\mathrm{nil}(R) = R\backslash U(R)$ and $\mathrm{vnr}(R) = U(R) \cup \{0\}$, in which case R is quasilocal with maximal ideal $\mathrm{nil}(R)$ (i.e., R is a zero-dimensional quasilocal ring). Equivalently, $R = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$ if and only if $\dim(R) = 0$ and R is either reduced or quasilocal. We also show that $R = \mathrm{vnr}(R) \cup Z(R)$ if and only if R is a total quotient ring.

Theorem 2.4. *Let R be a commutative ring.*

(1) $R = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$ *if and only if either R is von Neumann regular or R is quasilocal with maximal ideal* $\mathrm{nil}(R)$. *In particular, if* $R = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$, *then R is a π-regular ring, a von Neumann local ring, and a clean ring.*

(2) $R = \mathrm{vnr}(R) \cup Z(R)$ *if and only if* $T(R) = R$.

Proof. (1) Suppose that $R = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$. If $\mathrm{vnr}(R) = U(R) \cup \{0\}$, then $\mathrm{nil}(R)$ is the set of nonunits of R. So in this case, $\mathrm{nil}(R)$ is the unique maximal ideal of R. Thus we may assume that $\mathrm{vnr}(R)$ contains a nonzero nonunit, and hence there is $e \in \mathrm{Idem}(R)\backslash\{0,1\}$ by Theorem 2.1(6). We show that $\mathrm{nil}(R) = \{0\}$. Let $x \in \mathrm{nil}(R)$. Then necessarily $e + x \in \mathrm{vnr}(R)$, and thus $x - ex = (1-e)x = (1-e)(e+x) \in \mathrm{vnr}(R)$ by Theorem 2.1(2). Also, $x - ex = (1-e)x \in \mathrm{nil}(R)$; so $x - ex = 0$ by Theorem 2.1(3). By replacing e with $1 - e$, a similar argument yields $ex = 0$, and thus $x = 0$. Hence $R = \mathrm{vnr}(R) \cup \mathrm{nil}(R) = \mathrm{vnr}(R)$; so R is von Neumann regular.

For the converse, assume that R is either von Neumann regular or quasilocal with maximal ideal $\mathrm{nil}(R)$. If R is von Neumann regular, then $\mathrm{vnr}(R) = R$; so $R = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$. If R is quasilocal with maximal ideal $\mathrm{nil}(R)$, then $\mathrm{vnr}(R) = U(R) \cup \{0\} = (R\backslash\mathrm{nil}(R)) \cup \{0\}$ by Theorem 2.1(5); so again $R = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$.

For the "in particular" statement, suppose that $R = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$. Clearly $\mathrm{vnr}(R) \cup \mathrm{nil}(R) \subseteq \pi\text{-}r(R)$ (cf. Theorem 4.1(4)); so R is a π-regular ring when $R = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$. Note that if $a \in \mathrm{nil}(R)$, then $1 - a \in U(R) \subseteq \mathrm{vnr}(R)$ (cf.

Theorem 5.1(3)). Thus R is a von Neumann local ring when $R = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$. By Theorem 6.1(1)(2), a von Neumann local ring (or a π-regular ring) is also a clean ring.

(2) Note that $R = \mathrm{vnr}(R) \cup Z(R)$ if and only if $R = U(R) \cup Z(R)$ since $U(R) \subseteq \mathrm{vnr}(R) \subseteq U(R) \cup Z(R)$ by Theorem 2.1(4). Thus $R = \mathrm{vnr}(R) \cup Z(R)$ if and only if $T(R) = R$. ☐

If $Z(R) \subseteq \mathrm{vnr}(R)$, then $\mathrm{vnr}(R) = U(R) \cup Z(R)$ by Theorem 2.1(4). So if $Z(R) = \{0\}$, then R is an integral domain and $\mathrm{vnr}(R) = U(R) \cup \{0\}$. We next show that if $\{0\} \subsetneq Z(R) \subseteq \mathrm{vnr}(R)$, then R is von Neumann regular. One consequence of the next result is that to check if a non-domain R is von Neumann regular, we only need to show that each zero-divisor of R is von Neumann regular. Easy examples show that in Theorems 2.5 and 2.7 it is necessary to assume that $\{0\} \subsetneq Z(R)$.

Theorem 2.5. *Let R be a commutative ring with $\{0\} \subsetneq Z(R)$. Then $Z(R) \subseteq \mathrm{vnr}(R)$ if and only if R is von Neumann regular.*

Proof. Suppose that $\{0\} \subsetneq Z(R) \subseteq \mathrm{vnr}(R)$. Thus $\mathrm{vnr}(R)$ contains a nonzero nonunit; so there is $e \in \mathrm{Idem}(R)\backslash\{0,1\}$ by Theorem 2.1(6). Let $x \in R\backslash Z(R)$. Then $ex \in Z(R) \subseteq \mathrm{vnr}(R)$; so $(ex)^2 s = ex$ for some $s \in R$. Thus $exs = e$ since e is idempotent and x is not a zero-divisor. Similarly, $(1-e)xt = 1-e$ for some $t \in R$. Hence $(es + (1-e)t)x = exs + (1-e)xt = e + (1-e) = 1$; so $x \in U(R) \subseteq \mathrm{vnr}(R)$. Thus $R = \mathrm{vnr}(R)$, and hence R is von Neumann regular. The converse is clear. ☐

A Boolean ring is von Neumann regular; in fact, a von Neumann regular ring R is a Boolean ring if and only if $U(R) = \{1\}$ by Theorem 2.2. Clearly $\mathrm{Idem}(R)$ is multiplicatively closed, $\mathrm{Idem}(R) \cap \mathrm{nil}(R) = \{0\}$, $\mathrm{Idem}\big(\prod R_\alpha\big) = \prod \mathrm{Idem}(R_\alpha)$, and $f(\mathrm{Idem}(R)) \subseteq \mathrm{Idem}(S)$ for a homomorphism $f : R \to S$ of commutative rings. It is easily shown that $\mathrm{Idem}(R)$ is a subring of R if and only if $\mathrm{char}(R) = 2$ (and in this case, $\mathrm{Idem}(R)$ is a Boolean ring). The next two results are the analogs for idempotent elements of the two previous theorems (D.D. Anderson has shown us another proof of Theorem 2.7). The related result that $R = U(R) \cup \mathrm{Idem}(R)$ if and only if R is a Boolean ring or a field is given in [4, Theorem 1.14].

Theorem 2.6. *Let R be a commutative ring. Then $R = \mathrm{Idem}(R) \cup \mathrm{nil}(R)$ if and only if R is Boolean.*

Proof. Suppose that $R = \mathrm{Idem}(R) \cup \mathrm{nil}(R)$. Then $U(R) = \{1\}$ since $U(R) \subseteq \mathrm{Idem}(R)$; so we must have $\mathrm{nil}(R) = \{0\}$ since $U(R) + \mathrm{nil}(R) = U(R)$. Thus $R = \mathrm{Idem}(R)$, and hence R is Boolean. The converse is clear. ☐

Theorem 2.7. *Let R be a commutative ring with $\{0\} \subsetneq Z(R)$. Then $Z(R) \subseteq \mathrm{Idem}(R)$ if and only if R is Boolean.*

Proof. Suppose that $\{0\} \subsetneq Z(R) \subseteq \mathrm{Idem}(R)$; let $s \in Z(R)^*$. Since $s \in Z(R)^* \subseteq \mathrm{Idem}(R)$, we have $s, 1-s \in \mathrm{Idem}(R)$, and hence $1 - s \in Z(R)^*$. Let $x \in R\backslash Z(R)$. Then sx, $(1-s)x \in Z(R)^* \subseteq \mathrm{Idem}(R)$; so $(sx)^2 = sx$. Thus $sx = s$ since s is idempotent and x is not a zero-divisor. Similarly, $(1-s)x = 1-s$. Hence $x = sx + (1-s)x = s + (1-s) = 1$; so $x \in \mathrm{Idem}(R)$. Thus $R = \mathrm{Idem}(R)$, and hence

R is Boolean. The converse is clear. $\square$

It seems natural to conjecture that $R = \mathrm{Idem}(R) \cup Z(R)$ if and only if R is a Boolean ring. We next give some evidence to support this conjecture.

Theorem 2.8. *Let R be a commutative ring.*

(1) *If $R = \mathrm{Idem}(R) \cup Z(R)$, then $U(R) = \{1\}$, $\mathrm{char}(R) = 2$, $\mathrm{nil}(R) = \{0\}$, $J(R) = \{0\}$, and $T(R) = R$.*

(2) *If either $\dim(R) = 0$ or R has only a finite number of maximal ideals, then $R = \mathrm{Idem}(R) \cup Z(R)$ if and only if R is Boolean.*

Proof. (1) Suppose that $R = \mathrm{Idem}(R) \cup Z(R)$. Then $U(R) \subseteq \mathrm{Idem}(R)$; so $U(R) = \{1\}$. Since $U(R) = \{1\}$, we have $-1 = 1$; so $\mathrm{char}(R) = 2$. Since $U(R) + \mathrm{nil}(R) = U(R)$ and $U(R) = \{1\}$, necessarily $\mathrm{nil}(R) = \{0\}$. Similarly, $U(R) + J(R) = U(R)$ yields $J(R) = \{0\}$. Finally, $T(R) = R$ since $R = U(R) \cup Z(R)$.

(2) Suppose that $\dim(R) = 0$ and $R = \mathrm{Idem}(R) \cup Z(R)$. Then R is reduced by (1), and thus R is von Neumann regular by [19, Theorem 3.1, page 10]. Hence R is a von Neumann regular ring with $U(R) = \{1\}$ by (1); so R is a Boolean ring by Theorem 2.2. The converse is clear.

If R has only a finite number of maximal ideals and $R = \mathrm{Idem}(R) \cup Z(R)$, then R is isomorphic to a finite direct product of fields by the Chinese Remainder Theorem since $J(R) = \{0\}$ by (1). Thus R is a von Neumann regular ring, and hence a Boolean ring since $U(R) = \{1\}$ by (1). The converse is clear. $\square$

By Theorem 2.1(2)(4), $\mathrm{vnr}(R)$ is a multiplicatively closed subset of R with $\mathrm{Idem}(R) \cup U(R) \subseteq \mathrm{vnr}(R) \subseteq U(R) \cup Z(R)$. One can ask when $\mathrm{vnr}(R)$ is closed under addition, i.e., when is $\mathrm{vnr}(R)$ a subring of R? In Theorem 2.11, we answer this question when $2 \in U(R)$. We first show that $\mathrm{vnr}(R)$ a subring of R forces R to be reduced.

Theorem 2.9. *Let R be a commutative ring. If $\mathrm{vnr}(R)$ is a subring of R, then R is reduced.*

Proof. Let $x \in \mathrm{nil}(R)$. Then $1 + x \in U(R) \subseteq \mathrm{vnr}(R)$, and thus $x = -1 + (1 + x) \in \mathrm{vnr}(R)$ since $\mathrm{vnr}(R)$ is closed under addition. Hence $x \in \mathrm{nil}(R) \cap \mathrm{vnr}(R) = \{0\}$ by Theorem 2.1(3); so R is reduced. $\square$

It is well known that if R is a commutative von Neumann regular ring with $2 \in U(R)$, then every element of R is the sum of two units of R. In [15], it is shown that if $aua = a$ for some $u \in U(R)$, then a is the sum of two units of R. So this result extends to $\mathrm{vnr}(R)$.

Theorem 2.10. [15] *Let R be a commutative ring with $2 \in U(R)$. Then every $a \in \mathrm{vnr}(R)$ is the sum of two units of R.*

Proof. Let $a \in \mathrm{vnr}(R)$. Then $a = ue$ for some $u \in UR)$ and $e \in \mathrm{Idem}(R)$ by Theorem 2.2. Note that $(2e-1)^2 = 4e^2 - 4e + 1 = 1$; so $v = 2e - 1 \in U(R)$ with $v^2 = 1$. Thus $e = 2^{-1}v + 2^{-1}$; so $a = ue = u(2^{-1}v + 2^{-1}) = 2^{-1}uv + 2^{-1}u$ is the sum of two units of R. $\square$

Theorem 2.11. *Let R be a commutative ring with $2 \in U(R)$. Then the following statements are equivalent:*

(1) $\mathrm{vnr}(R)$ *is a subring of R.*
(2) *The sum of any four units of R is a von Neumann regular element of R.*
(3) *Let $u, v, k, m \in U(R)$ with $k^2 = m^2 = 1$. Then $u(1+k) + v(1+m) \in \mathrm{vnr}(R)$.*

Proof. (1)⇒(2) This is clear since $U(R) \subseteq \mathrm{vnr}(R)$ by Theorem 2.1(4).

(2)⇒(3) Let $u, v \in U(R)$ and $k, m \in U(R)$ with $k^2 = m^2 = 1$. Then we have $u(1+k) + v(1+m) = u + uk + v + vm$ is the sum of four units of R, and thus $u(1+k) + v(1+m) \in \mathrm{vnr}(R)$ by hypothesis.

(3)⇒(1) By Theorem 2.1(2)(4), we only need to show that $x, y \in \mathrm{vnr}(R)$ implies that $x + y \in \mathrm{vnr}(R)$. Let $x, y \in \mathrm{vnr}(R)$. Then $x = ue$ and $y = vf$ for some $u, v \in U(R)$ and $e, f \in \mathrm{Idem}(R)$ by Theorem 2.2. By the proof of Theorem 2.10, there are $k, m \in U(R)$ with $k^2 = m^2 = 1$ such that $2e = k + 1$ and $2f = m + 1$. Thus $2(x + y) = 2ue + 2vf = u(1 + k) + v(1 + m) \in \mathrm{vnr}(R)$. Since $2 \in U(R)$, we have $x + y \in \mathrm{vnr}(R)$ by Theorem 2.1(4). □

Remark 2.12. Let R be a commutative ring and $S(R) = \{x \in R \,|\, x^2 = 1\} \subseteq U(R)$. The proof of Theorem 2.10 shows that there is a map $\varphi : \mathrm{Idem}(R) \to S(R)$ given by $\varphi(e) = 2e - 1$. Note that φ is injective if $2 \notin Z(R)$ and φ is surjective, and thus bijective if $2 \in U(R)$.

3 Ring Extensions

In this section, we determine $\mathrm{vnr}(T)$ for several ring extensions T of R. We first determine $\mathrm{vnr}(R[X])$ and $\mathrm{vnr}(R[[X]])$. We have $\mathrm{vnr}(R) \subsetneq \mathrm{vnr}(R[X])$ when $U(R) \subsetneq U(R[X])$ (i.e., when R is not reduced), and we always have $\mathrm{vnr}(R[X]) \subsetneq \mathrm{vnr}(R[[X]])$ since $U(R[X]) \subsetneq U(R[[X]])$.

Lemma 3.1. *Let R be a commutative ring.*

(1) $U(R[X]) = \{ \sum a_n X^n \in R[X] \,|\, a_0 \in U(R),\ a_n \in \mathrm{nil}(R)$ *for every* $n \geq 1 \}$.
(2) $U(R[[X]]) = \{ \sum a_n X^n \in R[[X]] \,|\, a_0 \in U(R) \}$.
(3) $\mathrm{Idem}(R[[X]]) = \mathrm{Idem}(R[X]) = \mathrm{Idem}(R)$.

Proof. (1) and (2) are well known.

(3) Clearly $\mathrm{Idem}(R) \subseteq \mathrm{Idem}(R[X]) \subseteq \mathrm{Idem}(R[[X]])$. Thus we only need to show that each $f(X) = \sum a_n X^n \in \mathrm{Idem}(R[[X]])$ is actually in $\mathrm{Idem}(R)$. By comparing coefficients in $f(X)^2 = f(X)$, we have $a_0^2 = a_0$; so $a_0 \in \mathrm{Idem}(R)$. For $n = 1$, we have $2a_0a_1 = a_1$. Multiplying both sides by a_0 and using $a_0^2 = a_0$, we obtain $2a_0a_1 = a_0a_1$. Thus $a_0a_1 = 0$, and hence $a_1 = 2a_0a_1 = 0$. In a similar manner, one can easily show that $a_0^2 = a_0$ and $a_1 = \cdots = a_n = 0$ implies $a_{n+1} = 0$ for every $n \geq 1$. Thus $f(X) = a_0 \in \mathrm{Idem}(R)$. □

Theorem 3.2. *Let R be a commutative ring.*

(1) [2, Theorem 4.4] $\mathrm{vnr}(R[X]) = \{ \sum a_n X^n \in R[X] \mid a_0 = ue,\ a_n \in e(\mathrm{nil}(R))$ *for every* $n \geq 1$ *for some* $u \in U(R)$ *and* $e \in \mathrm{Idem}(R) \}$.
(2) [2, Theorem 4.5] $\mathrm{vnr}(R[[X]]) = \{ \sum a_n X^n \in R[[X]] \,|\, a_0 = ue,\ a_n \in eR$ *for every* $n \geq 1$ *for some* $u \in U(R)$ *and* $e \in \mathrm{Idem}(R) \}$.

(3) *$R[X]$ and $R[[X]]$ are never von Neumann regular rings. In particular, $R[X]$ and $R[[X]]$ are never Boolean rings.*

Proof. (1) and (2) follow directly from Theorem 2.2 and Lemma 3.1.

(3) This follows since X is not a von Neumann regular element in either ring. An alternate proof would be to note that $R[X]$ and $R[[X]]$ each have Krull dimension at least one. The "in particular" statement is clear. $\square$

Corollary 3.3. *Let R be a reduced commutative ring. Then* $\mathrm{vnr}(R) = \mathrm{vnr}(R[X]) \subsetneq \mathrm{vnr}(R[[X]])$.

In view of the preceding theorem, it is natural to ask if $\mathrm{vnr}(R[X]) = \{\sum a_n X^n \in R[X] \mid a_0 \in \mathrm{vnr}(R),\ a_n \in \mathrm{nil}(R) \text{ for every } n \geq 1\}$. The next example shows that this is not the case.

Example 3.4. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_4$ and $f(X) = (1,0) + (0,2)X \in R[X]$. Then $a_0 = (1,0) \in \mathrm{vnr}(R)$, $a_1 = (0,2) \in \mathrm{nil}(R)$, and it is easy to check that there are no $e \in \mathrm{Idem}(R)$ and $u \in U(R[X])$ such that $f(X) = ue$. Thus $f(X) \notin \mathrm{vnr}(R[X])$.

Does the converse of the "in particular" statement in Theorem 2.1(5) hold, i.e., does $\mathrm{vnr}(R) = U(R) \cup \{0\}$ imply that R is either an integral domain or quasilocal? We next show that the converse does not hold.

Example 3.5. Let R be a reduced quasilocal commutative ring which is not an integral domain. For example, let $R = K[X,Y]_{(X,Y)}/(XY)_{(X,Y)}$, where K is a field. Thus $\mathrm{vnr}(R) = U(R) \cup \{0\}$ by Theorem 2.1(5). Then $R[X]$ is reduced, and hence $U(R[X]) = U(R)$ by Lemma 3.1(1). Moreover, $R[X]$ is neither quasilocal nor an integral domain. By Corollary 3.3, $\mathrm{vnr}(R[X]) = \mathrm{vnr}(R) = U(R) \cup \{0\} = U(R[X]) \cup \{0\}$.

We next determine the von Neumann regular elements in an idealization. Given a commutative ring R and an R-module M, the *idealization* of M is the ring $R(+)M = R \times M$ with addition defined by $(r,m) + (s,n) = (r+s, m+n)$ and multiplication defined by $(r,m)(s,n) = (rs,\ rn+sm)$ for all $r,s \in R$ and $m,n \in M$. Note that $\{0\}(+)M \subseteq \mathrm{nil}(R(+)M)$ since $(\{0\}(+)M)^2 = \{(0,0)\}$. The following lemma records some useful facts about $R(+)M$. For other results about the ring $R(+)M$, see [5] and [19]. Note that $R(+)M$ is a Boolean ring if and only if R is a Boolean ring and $M = \{0\}$ by Lemma 3.6(4).

Lemma 3.6. *Let R be a commutative ring and M an R-module.*

(1) $U(R(+)M) = U(R)(+)M$.
(2) $\mathrm{nil}(R(+)M) = \mathrm{nil}(R)(+)M$.
(3) $Z(R(+)M) = \{(r,m) \mid r \in Z(R) \cup Z(M),\ m \in M\}$.
(4) $\mathrm{Idem}(R(+)M) = \mathrm{Idem}(R)(+)\{0\}$.
(5) $R(+)M/\{0\}(+)M \cong R$ *as rings.*

Proof. The proofs of (1), (2) and (3) may be found in [19, Theorem 25.1(6), page 162], [19, Theorem 25.1(5)] and [19, Theorem 25.3], respectively. (4) is from [5, Theorem 3.7], and (5) is from [5, Theorem 3.1]. $\square$

Theorem 3.7. *Let R be a commutative ring and M an R-module.*

(1) $\mathrm{vnr}(R(+)M) = \{(r, rm) \mid r \in \mathrm{vnr}(R),\, m \in M\}$.

(2) *$R(+)M$ is von Neumann regular if and only if R is von Neumann regular and $M = \{0\}$.*

Proof. (1) Let $(r, n) \in \mathrm{vnr}(R(+)M)$. Then $(r, n)^2(s, b) = (r, n)$ for some $(s, b) \in R(+)M$. Thus $(r, n) = (r^2, 2rn)(s, b) = (r^2s, r^2b + 2rsn)$; so $r = r^2s$ and $n = r^2b + rsn$. Hence $rsn = r^3sb + (rs)^2n = r^2b + rsn$. Thus $r^2b = 0$, and hence $rsn = n$. Thus $(r, n) = (r, rm)$ with $r \in \mathrm{vnr}(R)$ and $m = sn \in M$.

For the reverse inclusion, suppose that $r^2s = r$ for some $s \in R$, and let $(r, rm) \in R(+)M$. Then $(r, rm)^2(s, -sm) = (r^2, 2r^2m)(s, -sm) = (r^2s, -r^2sm + 2r^2sm) = (r, -rm + 2rm) = (r, rm)$; so $(r, rm) \in \mathrm{vnr}(R(+)M)$.

(2) This follows directly from (1). □

Corollary 3.8. *Let R be a von Neumann regular commutative ring and M an R-module. Then $\mathrm{vnr}(R(+)M) = \{(r, rm) \mid r \in R,\, m \in M\}$.*

Corollary 3.9. *Let R be either an integral domain or a quasilocal commutative ring, and let M be an R-module. Then $\mathrm{vnr}(R(+)M) = U(R)(+)M \cup \{(0, 0)\}$ $(= U(R(+)M) \cup \{(0, 0)\})$.*

Corollary 3.10. *Let R be a commutative ring and M a nonzero R-module. Then $R(+)M = \mathrm{vnr}(R(+)M) \cup \mathrm{nil}(R(+)M)$ if and only if R is quasilocal with maximal ideal $\mathrm{nil}(R)$.*

Proof. This follows directly from Theorems 3.7 and 2.4. □

Corollary 3.11. *Let R be a reduced commutative ring and M a nonzero R-module. Then $\mathrm{vnr}(R(+)M) = \mathrm{vnr}(R)^*(+)M \cup \{(0, 0)\}$ if and only if R is a field. In particular, $R(+)M = \mathrm{vnr}(R(+)M) \cup \mathrm{nil}(R(+)M)$ when R is a field.*

Let I be an ideal of a commutative ring R. As in [12], the *amalgamated duplication of the ring R along the ideal I* is defined to be the subring $R \bowtie I = \{(r, r + i) \mid r \in R,\, i \in I\}$ of $R \times R$. For other results about the ring $R \bowtie I$, see [12] or [13].

Theorem 3.12. *Let R be a commutative ring and I an ideal of R. Then $R \bowtie I$ is a von Neumann regular (resp., Boolean) ring if and only if R is a von Neumann regular (resp., Boolean) ring.*

Proof. By [12, Theorem 3.5], $R \bowtie I$ is reduced if and only if R is reduced, and $\dim(R \bowtie I) = \dim(R)$ by [12, Corollary 3.3]. Thus $R \bowtie I$ is von Neumann regular if and only if R is von Neumann regular. The "Boolean ring" statement is clear since $\mathrm{Idem}(R \bowtie I) = (R \bowtie I) \cap (\mathrm{Idem}(R) \times \mathrm{Idem}(R))$. □

4 π-Regular Elements

Recall that a commutative ring R is π-regular if and only if $\pi\text{-}r(R) = R$, if and only if $\dim(R) = 0$ [19, Theorem 3.1, page 10]. The first two theorems of this section are the analogs of Theorems 2.1 and 2.2 for π-regular elements. Example 3.5 shows that the converse of the "in particular" statement in Theorem 4.1(5) does not hold. Easy examples show that Theorem 4.1(7) does not extend to arbitrary direct products of commutative rings (cf. Theorem 2.1(7)). The proofs of the following results are routine and are thus left to the reader.

Theorem 4.1. *Let $R, R_1, \ldots, R_n$ and S be commutative rings.*

(1) $\mathrm{vnr}(R) \subseteq \pi\text{-}r(R)$. *In particular, a von Neumann regular ring is a π-regular ring.*
(2) *Let $a \in R$. If $a^{2n}x = a^n$ for some $x \in R$ and $n \geq 1$, then $a^n x \in \mathrm{Idem}(R)$.*
(3) *$\pi\text{-}r(R)$ is multiplicatively closed.*
(4) $\mathrm{vnr}(R) \cup \mathrm{nil}(R) \subseteq \pi\text{-}r(R) \subseteq U(R) \cup Z(R)$.
(5) *$\pi\text{-}r(R) = U(R) \cup \mathrm{nil}(R)$ if and only if $\mathrm{Idem}(R) = \{0, 1\}$. In particular, $\pi\text{-}r(R) = U(R) \cup \mathrm{nil}(R)$ if R is either an integral domain or quasilocal.*
(6) *$\pi\text{-}r(R)$ contains a non-nilpotent nonunit if and only if $\{0, 1\} \subsetneq \mathrm{Idem}(R)$.*
(7) *$\pi\text{-}r(R_1 \times \cdots \times R_n) = \pi\text{-}r(R_1) \times \cdots \times \pi\text{-}r(R_n)$. In particular, $R_1 \times \cdots \times R_n$ is π-regular if and only if each R_i is π-regular.*
(8) *Let $f : R \to S$ be a homomorphism of commutative rings. Then $f(\pi\text{-}r(R)) \subseteq \pi\text{-}r(S)$. In particular, $\pi\text{-}r(R) \subseteq \pi\text{-}r(S)$ when R is a subring of S, and any homomorphic image of a π-regular ring is π-regular.*

Theorem 4.2. *Let R be a commutative ring. Then the following statements are equivalent for $a \in R$:*

(1) $a \in \pi\text{-}r(R)$.
(2) $a^n \in \mathrm{vnr}(R)$ *for some* $n \geq 1$.
(3) $a^n = ue$ *for some* $u \in U(R)$, $e \in \mathrm{Idem}(R)$, *and* $n \geq 1$.
(4) $a = b + w$ *for some* $b \in \mathrm{vnr}(R)$ *and* $w \in \mathrm{nil}(R)$.
(5) $a = ue + w$ *for some* $u \in U(R)$, $e \in \mathrm{Idem}(R)$, *and* $w \in \mathrm{nil}(R)$.
(6) $a + \mathrm{nil}(R) \in \mathrm{vnr}(R/\mathrm{nil}(R))$.
(7) $a^n b = 0$ *for some* $b \in R$ *and* $n \geq 1$ *with* $a^n + b \in U(R)$.
(8) $ab \in \mathrm{nil}(R)$ *for some* $b \in R$ *with* $a + b \in U(R)$.

Proof. (1)⇔(2) This is clear.

(2)⇔(3) and (4)⇔(5) follow from Theorem 2.2.

(1)⇒(5) See [20, Theorem 13].

(4)⇒(6) This follows directly from Theorem 2.1(8).

(6)⇒(3) Since $a + \mathrm{nil}(R) \in \mathrm{vnr}(R/\mathrm{nil}(R))$, we have $a + \mathrm{nil}(R) = uf + \mathrm{nil}(R)$ for some $u \in U(R)$ and $f + \mathrm{nil}(R) \in \mathrm{Idem}(R/\mathrm{nil}(R))$ by Theorem 2.2. Since $f + \mathrm{nil}(R) \in \mathrm{Idem}(R/\mathrm{nil}(R))$, we have $e = f + h \in \mathrm{Idem}(R)$ for some $h \in \mathrm{nil}(R)$ by [25, Corollary, page 73]. Thus $a = ue + w$ for some $e \in \mathrm{Idem}(R)$, $u \in U(R)$, and $w \in \mathrm{nil}(R)$. Since e is idempotent and $w^n = 0$ for some $n \geq 1$, we have $a^n = (ue + w)^n = u^n e + nu^{n-1}ew + \cdots + nuew^{n-1} = (u^n + nu^{n-1}w + \cdots + nuw^{n-1})e$. Hence $v = u^n + nu^{n-1}w + \cdots + nuw^{n-1} \in U(R) + \mathrm{nil}(R) \subseteq U(R)$, and thus $a^n = ve$ with $v \in U(R)$ and $e \in \mathrm{Idem}(R)$.

(3)⇒(7) Suppose that $a^n = ue$ for some $u \in U(R)$, $e \in \mathrm{Idem}(R)$, and $n \geq 1$. Let $b = u(1-e)$. Then $a^n b = (ue)(u(1-e)) = 0$ and $a^n + b = ue + u(1-e) = u \in U(R)$.

(7)⇒(1) Suppose that $a^n b = 0$ for some $b \in R$ and $n \geq 1$ with $a^n + b = u \in U(R)$. Then $a^n u = a^n(a^n + b) = a^{2n} + a^n b = a^{2n}$. Thus $a^{2n}u^{-1} = (a^n u)u^{-1} = a^n$; so $a \in \pi\text{-}r(R)$.

(5)⇒(8) Let $a = ue + w$ for $u \in U(R)$, $e \in \mathrm{Idem}(R)$, and $w \in \mathrm{nil}(R)$, and let $b = u(1 - e)$. Then $ab = u(1 - e)w \in \mathrm{nil}(R)$ and $a + b = (ue + w) + u(1 - e) = (ue + u(1 - e)) + w = u + w \in U(R) + \mathrm{nil}(R) \subseteq U(R)$.

(8)⇒(7) Suppose that $ac \in \mathrm{nil}(R)$ for some $c \in R$ with $a + c \in U(R)$. Then $(ac)^n = 0$ for some $n \geq 1$ and $a^n + c^n = (a+c)^n + d(ac) \in U(R) + \mathrm{nil}(R) \subseteq U(R)$ for some $d \in R$. Let $b = c^n$. Then $a^n b = a^n c^n = (ac)^n = 0$ and so $a^n + b = a^n + c^n \in U(R)$. □

Corollary 4.3. *Let R be a commutative ring.*

(1) $\pi\text{-}r(R) = \mathrm{vnr}(R) + \mathrm{nil}(R)$.

(2) $\pi\text{-}r(R)/\mathrm{nil}(R) = \mathrm{vnr}(R/\mathrm{nil}(R))$.

(3) $\pi\text{-}r(R) = \mathrm{vnr}(R)$ *if and only if R is reduced.*

(4) [16, Theorem 3] *If $2 \in U(R)$, then every $a \in \pi\text{-}r(R)$ is the sum of two units of R.*

Proof. (1) This follows from the equivalence of (1) and (4) in Theorem 4.2.

(2) This follows from the equivalence of (1) and (6) in Theorem 4.2.

(3) This follows from (1) since $\mathrm{vnr}(R) \cap \mathrm{nil}(R) = \{0\}$ by Theorem 2.1(3).

(4) By (1), $a = x + w$ with $x \in \mathrm{vnr}(R)$ and $w \in \mathrm{nil}(R)$, and $x = u + v$ with $u, v \in U(R)$ by Theorem 2.10. Thus $a = u + (v + w)$ with $u, v + w \in U(R)$. □

By Theorem 4.1(4) and Corollary 4.3(1), we have $\mathrm{vnr}(R) \cup \mathrm{nil}(R) \subseteq \pi\text{-}r(R) = \mathrm{vnr}(R) + \mathrm{nil}(R)$. So it is natural to ask when $\pi\text{-}r(R) = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$. The next theorem, which is the π-regular analog of Theorem 2.4, shows that this happens only in the two extreme cases: either $\mathrm{vnr}(R) = U(R) \cup \{0\}$ (equivalently, $\mathrm{Idem}(R) = \{0, 1\}$ by Theorem 2.1(5)) or $\mathrm{nil}(R) = \{0\}$.

Theorem 4.4. *Let R be a commutative ring.*

(1) $\pi\text{-}r(R) = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$ *if and only if either* $\mathrm{Idem}(R) = \{0, 1\}$ *or* $\mathrm{nil}(R) = \{0\}$.

(2) $R = \pi\text{-}r(R) \cup Z(R)$ *if and only if* $T(R) = R$.

Proof. (1) Suppose that $\pi\text{-}r(R) = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$ and there is $e \in \mathrm{Idem}(R) \backslash \{0, 1\}$. We show that $\mathrm{nil}(R) = \{0\}$. Let $x \in \mathrm{nil}(R)$. Then $e + x \in \mathrm{vnr}(R) + \mathrm{nil}(R) = \pi\text{-}r(R) = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$ by Corollary 4.3(1) and hypothesis, and thus necessarily $e + x \in \mathrm{vnr}(R)$. Hence $x - ex = (1 - e)x = (1 - e)(e + x) \in \mathrm{vnr}(R)$ by Theorem 2.1(2). Also, $x - ex = (1 - e)x \in \mathrm{nil}(R)$; so $x - ex = 0$ by Theorem 2.1(3). By replacing e with $1 - e$, a similar argument yields $ex = 0$, and thus $x = 0$. Hence $\mathrm{nil}(R) = \{0\}$.

Conversely, suppose that either $\mathrm{Idem}(R) = \{0, 1\}$ or $\mathrm{nil}(R) = \{0\}$. Since $\pi\text{-}r(R) = \mathrm{vnr}(R) + \mathrm{nil}(R)$ by Corollary 4.3(1) and $U(R) + \mathrm{nil}(R) = U(R)$, either condition gives $\pi\text{-}r(R) = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$.

(2) Note that $R = \pi\text{-}r(R) \cup Z(R)$ if and only if $R = U(R) \cup Z(R)$ since $U(R) \subseteq \pi\text{-}r(R) \subseteq U(R) \cup Z(R)$ by Theorem 4.1(4). Thus $R = \pi\text{-}r(R) \cup Z(R)$ if and only if $T(R) = R$. □

If $Z(R) \subseteq \pi\text{-}r(R)$, then $\pi\text{-}r(R) = U(R) \cup Z(R)$ by Theorem 4.1(4). The next result is the π-regular analog of Theorem 2.5; it says that to determine if a ring R with $\mathrm{nil}(R) \subsetneq Z(R)$ is π-regular, we only need to check that the zero-divisors of R are all π-regular. Note that it is necessary to assume that $\text{nil}(R) \subsetneq Z(R)$ in Theorem 4.5 since $R = \mathbb{Z}(+)\mathbb{Z}$ satisfies $\{0\} \neq \text{nil}(R) = \{0\}(+)\mathbb{Z} = Z(R) \subseteq \pi\text{-}r(R) = \{0, -1, 1\}(+)\mathbb{Z}$ by Lemma 3.6(2)(3) and Theorem 4.7(1), but R is not π-regular.

Theorem 4.5. *Let R be a commutative ring with $\text{nil}(R) \subsetneq Z(R)$. Then $Z(R) \subseteq \pi\text{-}r(R)$ if and only if R is π-regular.*

Proof. If $Z(R) \subseteq \pi\text{-}r(R)$, since $\text{nil}(R) \subsetneq Z(R) \subseteq \pi\text{-}r(R)$, there is $e \in \text{Idem}(R)\backslash\{0, 1\}$ by Theorem 4.1(6). Since $\text{vnr}(R/\text{nil}(R)) = \{a+\text{nil}(R) \mid a \in \pi\text{-}r(R)\}$ by Theorem 4.2 and $e + \text{nil}(R) \in \text{vnr}(R/\text{nil}(R))$, we have $\emptyset \neq Z(R/\text{nil}(R))^* \subseteq \text{vnr}(R/\text{nil}(R))$. Thus $R/\text{nil}(R)$ is a von Neumann regular ring by Theorem 2.5. Hence R is a π-regular ring by Theorem 4.2 (also see [19, Theorem 3.1, page 10]). The converse is clear. $\square$

Theorem 4.6. *Let R be a commutative ring and M an R-module.*

(1) $\pi\text{-}r(R[X]) = \big\{ \sum a_n X^n \in R[X] \mid a_0 \in \pi\text{-}r(R),\ a_n \in \text{nil}(R)$ *for every* $n \geq 1\big\}$.

(2) $\pi\text{-}r(R[[X]]) = \big\{\sum a_n X^n \in R[[X]] \mid a_0 = ue + w$ *with* $u \in U(R),\ e \in \text{Idem}(R),\ w \in \text{nil}(R);\ \sum_{n \geq 1} a_n X^n \in eR[[X]] + \text{nil}(R[[X]])\big\}$.

(3) $R[X]$ *and* $R[[X]]$ *are never π-regular rings.*

Proof. (1) By Corollary 4.3(1) and Theorem 3.2(1), $\pi\text{-}r(R[X]) = \text{vnr}(R[X]) + \text{nil}(R[X]) = \big\{ \sum a_n X^n + \sum w_n X^n \in R[X] \mid a_0 = ue,\ a_n \in e(\text{nil}(R))$ for every $n \geq 1$, $w_n \in \text{nil}(R)$ for every $n \geq 0$ for some $u \in U(R)$ and $e \in \text{Idem}(R)\big\} = \big\{ \sum a_n X^n \in R[X] \mid a_0 = ue + w$ for some $u \in U(R)$, $e \in \text{Idem}(R)$, $w \in \text{nil}(R)$; $a_n \in \text{nil}(R)$ for every $n \geq 1\big\} = \big\{ \sum a_n X^n \in R[X] \mid a_0 \in \pi\text{-}r(R),\ a_n \in \text{nil}(R)$ for every $n \geq 1\big\}$.

(2) This follows from Corollary 4.3(1) and Theorem 3.2(2).

(3) This follows since X is not a π-regular element in either ring. An alternate proof would be to note that $R[X]$ and $R[[X]]$ each have Krull dimension at least one. $\square$

Theorem 4.7. *Let R be a commutative ring and M an R-module.*

(1) $\pi\text{-}r(R(+)M) = \{(r, m) \mid r \in \pi\text{-}r(R),\ m \in M\} = \pi\text{-}r(R)(+)M$.

(2) $R(+)M$ *is a π-regular ring if and only if R is a π-regular ring.*

Proof. (1) By Corollary 4.3(1), Theorem 3.7(1), and Lemma 3.6(2), $\pi\text{-}r(R(+)M) = \text{vnr}(R(+)M) + \text{nil}(R(+)M) = \{(r, rm) + (w, n) \mid r \in \text{vnr}(R),\ w \in \text{nil}(R),\ m, n \in M\} = \{(r, m) \mid r \in \pi\text{-}r(R),\ m \in M\} = \pi\text{-}r(R)(+)M$.

(2) This follows directly from (1). $\square$

Recall that $\text{nil}(R)$ is of *bounded index* n if n is the least positive integer such that $w^n = 0$ for every $w \in \text{nil}(R)$. A commutative ring R is said to be of *bounded index* n if n is the least positive integer such that $a^n \in \text{vnr}(R)$ for every $a \in \pi\text{-}r(R)$ (cf. [24, page 332]). Note that a von Neumann regular ring is of bounded index 1.

Theorem 4.8. *Let R be a commutative ring and n a positive integer. Then R is of bounded index n if and only if $\text{nil}(R)$ is of bounded index n.*

Proof. Suppose that R is of bounded index n, and let $w \in \text{nil}(R) \subseteq \pi\text{-}r(R)$. Then $w^n \in \text{vnr}(R) \cap \text{nil}(R) = \{0\}$ by Theorem 2.1(3); so $w^n = 0$. Thus $\text{nil}(R)$ is of bounded index at most n.

Conversely, suppose that $\mathrm{nil}(R)$ is of bounded index n. Let $a \in \pi\text{-}r(R)$; so $a = ue + w$ for some $u \in U(R)$, $e \in \mathrm{Idem}(R)$, and $w \in \mathrm{nil}(R)$ by Theorem 4.2. The proof of (6)$\Rightarrow$(3) of Theorem 4.2 gives $a^n = ve$ for some $v \in U(R)$, and thus $a^n \in \mathrm{vnr}(R)$ by Theorem 2.2. Hence R has bounded index at most n, and thus $\mathrm{nil}(R)$ and R each have bounded index n. ☐

Corollary 4.9. *Let R be a commutative (resp., π-regular) ring of bounded index n, and let M be an R-module. Then $R(+)M$ is a commutative (resp., π-regular) ring of bounded index at most $n + 1$. In particular, if R is a von Neumann regular ring, then $R(+)M$ is a π-regular ring of bounded index at most 2.*

Proof. Note that $T = R(+)M$ is π-regular if R is π-regular by Theorem 4.7(2) and $\mathrm{nil}(T) = \mathrm{nil}(R)(+)M$ by Theorem 3.6(2). By Theorem 4.8, it suffices to show that $\mathrm{nil}(T)$ is of bounded index at most $n + 1$. Let $x = (w, m) \in \mathrm{nil}(T)$, where $w \in \mathrm{nil}(R)$ and $m \in M$. Since $\mathrm{nil}(R)$ is of bounded index n, we have $x^{n+1} = (w, m)^{n+1} = (w^{n+1}, (n + 1)w^n m) = (0, 0)$. Thus T is a commutative (resp., π-regular) ring of bounded index at most $n + 1$. The "in particular" statement is clear. ☐

Note that the π-regular rings R and $R(+)M$ may both have bounded index n even when M is nonzero. For example, let $R = \mathbb{Z}_2[X]/(X^2)$. Then R and $R(+)R$ are both π-regular rings and both have bounded index 2 since $\mathrm{char}(R) = 2$.

Theorem 4.10. *Let R be a commutative ring and I an ideal of R. Then $R \bowtie I$ is a π-regular ring if and only if R is a π-regular ring.*

Proof. Since a commutative ring is π-regular if and only if it is zero-dimensional [19, Theorem 3.1, page 10], the theorem follows from the fact that $\dim(R \bowtie I) = \dim(R)$ [12, Corollary 3.3]. ☐

5 Von Neumann Local Rings

Von Neumann local rings were introduced in [11] and have been further studied in [1] and [2].

Theorem 5.1. *Let R and S be commutative rings, and let $\{R_\alpha\}_{\alpha\in\Lambda}$ be a family of commutative rings.*

(1) $\mathrm{vnl}(R) = \mathrm{vnr}(R) \cup (1 + \mathrm{vnr}(R)) = \{0, 1\} + \mathrm{vnr}(R)$. *In particular,* $\{0, 1\} + U(R) = U(R) \cup (1 + U(R)) \subseteq \mathrm{vnl}(R)$.

(2) *Let $a \in R$. Then $a \in \mathrm{vnl}(R)$ if and only if there are $u \in U(R)$ and $e \in \mathrm{Idem}(R)$ such that either $a = ue$ or $a = 1 + ue$.*

(3) $\mathrm{nil}(R) \subseteq J(R) \subseteq \mathrm{vnl}(R)$. *Thus* $U(R) \cup J(R) \subseteq \mathrm{vnl}(R)$.

(4) $\mathrm{vnl}(R) = U(R) \cup (1 + U(R))$ *if and only if* $\mathrm{Idem}(R) = \{0, 1\}$. *In particular,* $\mathrm{vnl}(R) = U(R) \cup (1 + U(R))$ *when R is either an integral domain or quasilocal (note that* $\mathrm{vnl}(R) = R$ *when R is qusailocal).*

(5) *If* $\mathrm{vnl}(R) = \mathrm{vnr}(R)$, *then R is reduced,* $\pi\text{-}r(R) = \mathrm{vnr}(R)$, *and* $\mathbb{Z}1 + \mathrm{vnr}(R) = \mathrm{vnr}(R)$.

(6) $\mathrm{vnl}\big(\prod R_\alpha\big) \subseteq \prod \mathrm{vnl}(R_\alpha)$. *If* $\mathrm{vnl}\big(\prod R_\alpha\big)$ *is multiplicatively closed, then we have* $\mathrm{vnl}\big(\prod R_\alpha\big) = \prod \mathrm{vnl}(R_\alpha)$.

(7) [1, Theorem 3.1] $\mathrm{vnl}\big(\prod R_\alpha\big) = \prod \mathrm{vnl}(R_\alpha)$ *if and only if* $\mathrm{vnl}(R_\alpha) = \mathrm{vnr}(R_\alpha)$ *for all but at most one* α. *In particular,* $\prod R_\alpha$ *is a von Neumann local ring if and only if there is at most one* α *such that* R_α *is not von Neumann regular, and that* R_α *is von Neumann local.*

(8) *Let* $f : R \to S$ *be a homomorphism of commutative rings. Then* $f(\mathrm{vnl}(R)) \subseteq \mathrm{vnl}(S)$. *In particular,* $\mathrm{vnl}(R) \subseteq \mathrm{vnl}(S)$ *when* R *is a subring of* S, *and any homomorphic image of a von Neumann local ring is a von Neumann local ring.*

(9) [1, Theorem 3.8] *If* $2 \in U(R)$, *then every* $a \in \mathrm{vnl}(R)$ *is the sum of three units of* R.

Proof. (1) Since $1 - a \in \mathrm{vnr}(R) \Leftrightarrow a - 1 \in \mathrm{vnr}(R) \Leftrightarrow a \in 1 + \mathrm{vnr}(R)$, we have $\mathrm{vnl}(R) = \mathrm{vnr}(R) \cup (1 + \mathrm{vnr}(R))$. The "in particular" statement is clear since $U(R) \subseteq \mathrm{vnr}(R)$.

(2) This follows from (1) and Theorem 2.2.

(3) Clearly $\mathrm{nil}(R) \subseteq J(R)$. Let $a \in J(R)$. Then $1 - a \in U(R)$, and hence $a - 1 \in U(R)$. Thus $a \in 1 + U(R)$, and hence $\mathrm{nil}(R) \subseteq J(R) \subseteq 1 + U(R) \subseteq \mathrm{vnl}(R)$ by (1). Since $U(R) \subseteq \mathrm{vnl}(R)$, we have $U(R) \cup J(R) \subseteq \mathrm{vnl}(R)$.

(4) ($\Rightarrow$) Suppose that $\mathrm{vnl}(R) = U(R) \cup (1 + U(R))$. Let $e \in \mathrm{Idem}(R)$. If $e \in U(R)$, then $e = 1$. If $e \in 1 + U(R)$, then $1 - e \in U(R)$; so $1 - e = 1$, and hence $e = 0$. Thus $\mathrm{Idem}(R) = \{0, 1\}$.

($\Leftarrow$) Suppose that $\mathrm{Idem}(R) = \{0, 1\}$. Then $\mathrm{vnr}(R) = U(R) \cup \{0\}$ by Theorem 2.1(5). Thus $\mathrm{vnl}(R) = \mathrm{vnr}(R) \cup (1 + \mathrm{vnr}(R)) = U(R) \cup (1 + U(R))$ by (1).

The "in particular" statement follows since $\mathrm{Idem}(R) = \{0, 1\}$ when R is either an integral domain or quasilocal.

(5) Suppose that $\mathrm{vnl}(R) = \mathrm{vnr}(R)$. Then $\mathrm{nil}(R) \subseteq \mathrm{vnl}(R) = \mathrm{vnr}(R)$ by (3) and $\mathrm{vnr}(R) \cap \mathrm{nil}(R) = \{0\}$ by Theorem 2.1(3); so $\mathrm{nil}(R) = \{0\}$. Thus $\pi\text{-}r(R) = \mathrm{vnr}(R)$ by Corollary 4.3(3). Finally, if $\mathrm{vnl}(R) = \mathrm{vnr}(R)$, then $1 + \mathrm{vnr}(R) \subseteq \mathrm{vnr}(R)$ by (1). Hence $n1 + \mathrm{vnr}(R) \subseteq \mathrm{vnr}(R)$ for every $n \geq 0$, and thus $\mathbb{Z}1 + \mathrm{vnr}(R) \subseteq \mathrm{vnr}(R)$ since $-\mathrm{vnr}(R) = \mathrm{vnr}(R)$. Hence $\mathbb{Z}1 + \mathrm{vnr}(R) = \mathrm{vnr}(R)$.

(6) By (1) and Theorem 2.1(7), $\mathrm{vnl}\big(\prod R_\alpha\big) = \mathrm{vnr}\big(\prod R_\alpha\big) \cup \big(1 + \mathrm{vnr}\big(\prod R_\alpha\big)\big) = \prod \mathrm{vnr}(R_\alpha) \cup \big(1 + \prod \mathrm{vnr}(R_\alpha)\big) = \prod \mathrm{vnr}(R_\alpha) \cup \prod(1 + \mathrm{vnr}(R_\alpha)) \subseteq \prod \mathrm{vnl}(R_\alpha) \cup \prod \mathrm{vnl}(R_\alpha) = \prod \mathrm{vnl}(R_\alpha)$.

Suppose that $\mathrm{vnl}\big(\prod R_\alpha\big)$ is multiplicatively closed. Let $(r_\alpha) \in \prod_{\alpha \in \Lambda} \mathrm{vnl}(R_\alpha)$. Let $X = \{\alpha \in \Lambda \mid r_\alpha \in \mathrm{vnr}(R_\alpha)\}$; so $r_\alpha = 1 + s_\alpha \in 1 + \mathrm{vnr}(R_\alpha)$ for $\alpha \in \Lambda \backslash X$. Define (a_α) by $a_\alpha = r_\alpha$ for $\alpha \in X$ and $a_\alpha = 1$ for $\alpha \in \Lambda \backslash X$. Then $(a_\alpha) \in \prod \mathrm{vnr}(R_\alpha) = \mathrm{vnr}\big(\prod R_\alpha\big) \subseteq \mathrm{vnl}\big(\prod R_\alpha\big)$. Next, we define (b_α) by $b_\alpha = 1$ for $\alpha \in X$ and $b_\alpha = r_\alpha = 1 + s_\alpha$ for $\alpha \in \Lambda \backslash X$. Then $(b_\alpha) \in \prod(1 + \mathrm{vnr}(R_\alpha)) = 1 + \prod \mathrm{vnr}(R_\alpha) = 1 + \mathrm{vnr}\big(\prod R_\alpha\big) \subseteq \mathrm{vnl}\big(\prod R_\alpha\big)$. Thus $(r_\alpha) = (a_\alpha)(b_\alpha) \in \mathrm{vnl}\big(\prod R_\alpha\big)\mathrm{vnl}\big(\prod R_\alpha\big) \subseteq \mathrm{vnl}\big(\prod R_\alpha\big)$ since $\mathrm{vnl}\big(\prod R_\alpha\big)$ is multiplicatively closed. Hence $\prod \mathrm{vnl}(R_\alpha) \subseteq \mathrm{vnl}\big(\prod R_\alpha\big)$; so we have equality.

(7) ($\Rightarrow$) Suppose that $\mathrm{vnr}(R_\beta) \subsetneq \mathrm{vnl}(R_\beta)$ and $\mathrm{vnr}(R_\gamma) \subsetneq \mathrm{vnl}(R_\gamma)$ for distinct $\beta, \gamma \in \Lambda$. Then we have $a_\beta \in \mathrm{vnl}(R_\beta) \backslash \mathrm{vnr}(R_\beta)$ and $b_\gamma \in \mathrm{vnl}(R_\gamma) \backslash \mathrm{vnr}(R_\gamma)$. Thus $a_\gamma = 1 - b_\gamma \in \mathrm{vnr}(R_\gamma) \subseteq \mathrm{vnl}(R_\gamma)$. Let $a_\alpha = 1$ for all $\alpha \in \Lambda \backslash \{\beta, \gamma\}$. Then $(a_\alpha) \in \prod_{\alpha \in \Lambda} \mathrm{vnl}(R_\alpha)$. However, $(a_\alpha) \notin \prod_{\alpha \in \Lambda} \mathrm{vnr}(R_\alpha) = \mathrm{vnr}\big(\prod_{\alpha \in \Lambda} R_\alpha\big)$ since

$a_\beta \notin \mathrm{vnr}(R_\beta)$, and $(1-a_\alpha) \notin \prod_{\alpha\in\Lambda} \mathrm{vnr}(R_\alpha) = \mathrm{vnr}\big(\prod_{\alpha\in\Lambda} R_\alpha\big)$ since $1-a_\gamma = b_\gamma \notin \mathrm{vnr}(R_\gamma)$. Thus $(a_\alpha) \notin \mathrm{vnl}\big(\prod_{\alpha\in\Lambda} R_\alpha\big)$; so $\mathrm{vnl}\big(\prod R_\alpha\big) \subsetneq \prod \mathrm{vnl}(R_\alpha)$.

($\Leftarrow$) The " $\subseteq$ " inclusion follows from (6). For the reverse inclusion, let $(a_\alpha) \in \prod \mathrm{vnl}(R_\alpha)$. If $\mathrm{vnl}(R_\alpha) = \mathrm{vnr}(R_\alpha)$ for each $\alpha \in \Lambda$, then $(a_\alpha) \in \prod \mathrm{vnr}(R_\alpha) = \mathrm{vnr}\big(\prod R_\alpha\big) \subseteq \mathrm{vnl}\big(\prod R_\alpha\big)$. So by hypothesis, we may assume that there is only one $\beta \in \Lambda$ with $\mathrm{vnr}(R_\beta) \subsetneq \mathrm{vnl}(R_\beta)$ and that $1-a_\beta \in \mathrm{vnr}(R_\beta)$. In this case, $1-a_\alpha \in \mathrm{vnr}(R_\alpha)$ for all $\alpha \in \Lambda\backslash\{\beta\}$ by (5); so $(1-a_\alpha) \in \prod \mathrm{vnr}(R_\alpha) = \mathrm{vnr}\big(\prod R_\alpha\big)$. Thus $(a_\alpha) \in \mathrm{vnl}\big(\prod R_\alpha\big)$, and hence $\prod \mathrm{vnl}(R_\alpha) \subseteq \mathrm{vnl}\big(\prod R_\alpha\big)$.

The "in particular" statement is clear.

(8) Since f is a homomorphism, $f(\mathrm{vnl}(R)) = f(\mathrm{vnr}(R) \cup (1+\mathrm{vnr}(R))) \subseteq f(\mathrm{vnr}(R)) \cup f(1+\mathrm{vnr}(R)) \subseteq \mathrm{vnr}(S) \cup (1+\mathrm{vnr}(S)) = \mathrm{vnl}(S)$ by (1) and Theorem 2.1(8). The "in particular" statement is clear.

(9) This follows directly from Theorem 2.10 and (1). □

It follows from Theorem 5.1(7) that the inclusion $\mathrm{vnl}\big(\prod R_\alpha\big) \subseteq \prod \mathrm{vnl}(R_\alpha)$ in Theorem 5.1(6) may be proper. For example, $\mathbb{Z}_4$ is a von Neumann local ring, but $\mathbb{Z}_4 \times \mathbb{Z}_4$ is not von Neumann local (see Section 3 of [1]). Also, unlike $\mathrm{Idem}(R)$, $\mathrm{vnr}(R)$, and π-$r(R)$, the set $\mathrm{vnl}(R)$ need not be multiplicatively closed since $\mathrm{vnl}(\mathbb{Z}) = U(\mathbb{Z}) \cup (1+U(\mathbb{Z})) = \{-1,0,1,2\}$ by Theorem 5.1(4) (cf. Theorems 5.1(6) and 6.1(8)).

We next determine the von Neumann local elements in $R[X]$, $R[[X]]$, and $R(+)M$. Several other equivalent conditions for $R[[X]]$ to be a von Neumann local ring are given in [1, Theorem 4.6].

Theorem 5.2. *Let R be a commutative ring.*

(1) $\mathrm{vnl}(R[X]) = \big\{\sum a_n X^n \in R[X] \mid$ *either* $a_0 = ue$ *or* $a_0 = 1-ue$, *and* $a_n \in e(\mathrm{nil}(R))$ *for every* $n \geq 1$ *for some* $u \in U(R)$ *and* $e \in \mathrm{Idem}(R)\big\}$.

(2) [1, Corollary 4.8] $R[X]$ *is never a von Neumann local ring.*

(3) $\mathrm{vnl}(R[[X]]) = \big\{\sum a_n X^n \in R[[X]] \mid$ *either* $a_0 = ue$ *or* $a_0 = 1-ue$, *and* $a_n \in eR$ *for every* $n \geq 1$ *for some* $u \in U(R)$ *and* $e \in \mathrm{Idem}(R)\big\}$.

(4) [1, Theorem 4.6] $R[[X]]$ *is a von Neumann local ring if and only if R is a quasilocal ring.*

Proof. (1) This follows from Theorems 3.2(1) and 5.1(2).

(2) It follows from (1) that $X \notin \mathrm{vnl}(R[X])$; so $R[X]$ is never von Neumann local.

(3) This follows from Theorems 3.2(2) and 5.1(2).

(4) If R is quasilocal, then $R[[X]]$ is also quasilocal, and hence $R[[X]]$ is von Neumann local. Conversely, suppose that R is not quasilocal. Then $1 = a+b$ for nonzero nonunits $a, b \in R$. Thus neither $f = a+X$ nor $1-f = b-X$ is in $\mathrm{vnr}(R[[X]])$ by Theorem 3.2(2), and hence $f \notin \mathrm{vnl}(R[[X]])$; so $R[[X]]$ is not a von Neumann local ring. □

For the next theorem, observe that if $\mathrm{Idem}(R) = \{0,1\}$, then $\mathrm{vnl}(R) = \mathrm{cln}(R)$ (cf. Theorem 6.1(4), its proof is independent of earlier results). Thus if $\mathrm{Idem}(R) = \{0,1\}$, then R is a von Neumann local ring if and only R is a clean ring.

Theorem 5.3. *Let R be a commutative ring and M an R-module.*

(1) $\mathrm{vnl}(R(+)M) = \{(r, rm) \mid r \in \mathrm{vnr}(R),\ m \in M\} \cup \{(1+r, rm) \mid r \in \mathrm{vnr}(R),\ m \in M\}$.
(2) *R is a von Neumann local ring when $R(+)M$ is a von Neumann local ring.*
(3) *Suppose that there is $m \in M$ with $\mathrm{ann}_R(m) = \{0\}$. Then $R(+)M$ is a von Neumann local ring if and only if R is a von Neumann local ring with* $\mathrm{Idem}(R) = \{0,1\}$.
(4) *If M is a ring extension of R, then $R(+)M$ is a von Neumann local ring if and only if R is a von Neumann local ring with* $\mathrm{Idem}(R) = \{0,1\}$.

Proof. (1) Since the identity element of $R(+)M$ is $(1,0)$, we have $\mathrm{vnl}(R(+)M) = \mathrm{vnr}(R(+)M) \cup ((1,0) + \mathrm{vnr}(R(+)M)) = \{(r, rm) \mid r \in \mathrm{vnr}(R),\ m \in M\} \cup \{(1+r, rm) \mid r \in \mathrm{vnr}(R),\ m \in M\}$ by Theorems 5.1(1) and 3.7.

(2) This follows from Theorem 5.1(8) and Lemma 3.6(5).

(3) Suppose that R is a von Neumann local ring with $\mathrm{Idem}(R) = \{0,1\}$. Then R is a clean ring by Theorem 6.1(4). Thus $R(+)M$ is a clean ring with $\mathrm{Idem}(R(+)M) = \{0,1\}$ by Theorem 6.1(4) and Lemma 3.6(4), and hence $R(+)M$ is also a von Neumann local ring by Theorem 6.1(4) again.

Conversely, suppose that $R(+)M$ is a von Neumann local ring and $m \in M$ with $\mathrm{ann}_R(m) = \{0\}$. Then R is a von Neumann local ring by (2). Suppose that there is $e \in \mathrm{Idem}(R) \backslash \{0,1\}$. Since $(e, m) \in \mathrm{vnl}(R(+)M)$, either $(e,m) = (f_1, 0)(u, t)$ for some $f_1 \in \mathrm{Idem}(R) \backslash \{0,1\}$, $u \in U(R)$, and $t \in M$, or $(e,m) = (1,0) + (f_2, 0)(v, k)$ for some $f_2 \in \mathrm{Idem}(R) \backslash \{0,1\}$, $v \in U(R)$, and $k \in M$ by Theorem 5.1(2) and Lemma 3.6 (note that $f_2 \notin \{0,1\}$ since $1 + v \notin \mathrm{Idem}(R)$). In the first case, we have $(1 - f_1)m = 0$, and in the second case, we have $(1 - f_2)m = 0$, which are both contradictions since $\mathrm{ann}_R(m) = \{0\}$. Thus $\mathrm{Idem}(R) = \{0,1\}$.

(4) Suppose that M is a ring extension of R. Since $1 \in M$ and $\mathrm{ann}_R(1) = \{0\}$, the claim follows from (3). ☐

The next example shows that the hypothesis $\mathrm{Idem}(R) = \{0,1\}$ is needed in (3) and (4) of the above theorem.

Example 5.4. (a) Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, $M = R$, and $T = R(+)M$. Then T is a clean ring by Theorem 6.1(4) since R is von Neumann regular, and thus clean. However, T is not a von Neumann local ring by Theorem 5.3(4) since $\{0,1\} \subsetneq \mathrm{Idem}(R)$.

(b) Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, $M = \mathbb{Z}_2 \times \{0\}$, and $T = R(+)M$. Then R is a von Neumann regular ring, and thus a von Neumann local ring and a clean ring, but T is not a von Neumann local ring by Theorem 5.3(1).

6 Clean Elements

We first collect some elementary results about clean elements. In particular, we show that $\mathrm{vnl}(R) \subseteq \mathrm{cln}(R)$ (cf. [4, page 3331]) and $\pi\text{-}r(R) \subseteq \mathrm{cln}(R)$.

Theorem 6.1. *Let R and S be commutative rings, and let $\{R_\alpha\}$ be a family of commutative rings.*
(1) $\mathrm{Idem}(R) \subseteq \mathrm{vnr}(R) \subseteq \mathrm{vnl}(R) \subseteq \mathrm{cln}(R)$. *In particular, a Boolean ring, a von Neumann regular ring, or a von Neumann local ring is a clean ring.*
(2) $\mathrm{vnr}(R) \subseteq \pi\text{-}r(R) \subseteq \mathrm{cln}(R)$. *In particular, a π-regular ring is a clean ring.*

(3) *$U(R) \cup J(R) \subseteq U(R) \cup (1 + U(R)) \subseteq \mathrm{cln}(R)$.*
(4) *If $\mathrm{Idem}(R) = \{0, 1\}$, then $\mathrm{cln}(R) = \mathrm{vnl}(R)$. In particular, $\mathrm{cln}(R) = \mathrm{vnl}(R)$ when R is either an integral domain or quasilocal (note that $\mathrm{cln}(R) = \mathrm{vnl}(R)$ $= R$ when R is quasilocal).*
(5) [18, Proposition 7] *$\mathrm{cln}\big(\prod R_\alpha\big) = \prod \mathrm{cln}(R_\alpha)$. In particular, $\prod R_\alpha$ is a clean ring if and only if each R_α is a clean ring.*
(6) *Let $f : R \to S$ be a homomorphism of commutative rings. Then $f(\mathrm{cln}(R)) \subseteq \mathrm{cln}(S)$. In particular, $\mathrm{cln}(R) \subseteq \mathrm{cln}(S)$ when R is a subring of S, and any homomorphic image of a clean ring is a clean ring.*
(7) *If $2 \in U(R)$, then every $a \in \mathrm{cln}(R)$ is the sum of three units of R.*
(8) *If $\mathrm{vnl}(R)$ is multiplicatively closed, then $\mathrm{cln}(R) = \mathrm{vnl}(R)$.*

Proof. (1) We first show that $\mathrm{vnr}(R) \subseteq \mathrm{cln}(R)$. Let $a \in \mathrm{vnr}(R)$. Then $a = ue$ for some $u \in U(R)$ and $e \in \mathrm{Idem}(R)$ by Theorem 2.2. Thus $a = (ue + e - 1) + (1 - e)$ with $ue + e - 1 \in U(R)$ since $(ue + e - 1)(u^{-1}e + e - 1) = 1$ and $1 - e \in \mathrm{Idem}(R)$; so $a \in \mathrm{cln}(R)$. We next show that $1 + \mathrm{vnr}(R) \subseteq \mathrm{cln}(R)$. Let $a = ue$ as above. Then $1+a = 1+ue = (ue+1-e)+e$ with $ue+1-e \in U(R)$ since $(ue+1-e)(u^{-1}e+1-e)$ $= 1$ and $e \in \mathrm{Idem}(R)$; so $1+\mathrm{vnr}(R) \subseteq \mathrm{cln}(R)$. Hence $\mathrm{vnl}(R) = \mathrm{vnr}(R)\cup(1+\mathrm{vnr}(R))$ $\subseteq \mathrm{cln}(R)$. The "in particular" statement is clear.

(2) Let $x \in \pi\text{-}r(R)$. Then $x = a + w$ for some $a \in \mathrm{vnr}(R)$ and $w \in \mathrm{nil}(R)$ by Theorem 4.2. Since $\mathrm{vnr}(R) \subseteq \mathrm{vnl}(R) \subseteq \mathrm{cln}(R)$ by (1), we have $a = u + e$ for some $u \in U(R)$ and $e \in \mathrm{Idem}(R)$. Thus $x = a + w = u + e + w = (u + w) + e \in U(R) + \mathrm{Idem}(R) = \mathrm{cln}(R)$. The "in particular" statement is clear.

(3) This follows from (1) and Theorem 5.1(1)(3).

(4) Suppose that $\mathrm{Idem}(R) = \{0, 1\}$. Then $\mathrm{cln}(R) = \{0, 1\} + U(R) = U(R) \cup (1 + U(R)) \subseteq \mathrm{vnl}(R)$; so $\mathrm{cln}(R) = \mathrm{vnl}(R)$. The "in particular" statement is clear since $\mathrm{Idem}(R) = \{0, 1\}$ when R is either an integral domain or quasilocal.

(5) This is clear.

(6) Since $f(\mathrm{Idem}(R)) \subseteq \mathrm{Idem}(S)$, $f(U(R)) \subseteq U(S)$, and f is a homomorphism, we have $f(\mathrm{cln}(R)) = f(U(R) + \mathrm{Idem}(R)) = f(U(R)) + f(\mathrm{Idem}(R)) \subseteq U(S) + \mathrm{Idem}(S) = \mathrm{cln}(S)$. The "in particular" statement is clear.

(7) This follows from Theorem 2.10 since $\mathrm{Idem}(R) \subseteq \mathrm{vnr}(R)$.

(8) By (1), we always have $\mathrm{vnl}(R) \subseteq \mathrm{cln}(R)$. Now suppose that $\mathrm{vnl}(R)$ is multiplicatively closed. Let $x = e + u \in \mathrm{cln}(R)$ with $e \in \mathrm{Idem}(R)$ and $u \in U(R)$. Then $x = e + u = u(u^{-1}e + 1) \in \mathrm{vnl}(R)$ since $u \in U(R) \subseteq \mathrm{vnl}(R)$ by Theorem 5.1(3), $u^{-1}e + 1 \in 1 + \mathrm{vnr}(R) \subseteq \mathrm{vnl}(R)$ by Theorems 2.2 and 5.1(1), and $\mathrm{vnl}(R)$ is multiplicatively closed. Thus $\mathrm{cln}(R) = \mathrm{vnl}(R)$. □

Like $\mathrm{vnl}(R)$, the set $\mathrm{cln}(R)$ need not be multiplicatively closed since $\mathrm{cln}(\mathbb{Z}) = \mathrm{vnl}(\mathbb{Z}) = \{-1, 0, 1, 2\}$ by Theorem 6.1(4) (this example also shows that the converse of Theorem 6.1(8) does not hold). However, parts (1), (5), and (8) of Theorem 6.1 may be used to give another proof of the "if, then" statement in Theorem 5.1(6). Also, the converse of Theorem 6.1(4) is false since $\mathrm{cln}(R) = \mathrm{vnl}(R) = \mathrm{Idem}(R) = R$ for any Boolean ring R. The next example shows that the inclusions between $\mathrm{Idem}(R)$, $\mathrm{vnr}(R)$, $\pi\text{-}r(R)$, $\mathrm{vnl}(R)$, and $\mathrm{cln}(R)$ in Theorem 6.1(1)(2) may all be strict.

Example 6.2. Let $R = \mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_4$. Then $\mathrm{Idem}(R) = \{0,1\}\times\{0,1\}\times\{0,1\}$, $\mathrm{vnr}(R) = \{-1,0,1\}\times\{0,1,3\}\times\{0,1,3\}$ by Theorem 2.1(7), and $\pi\text{-}r(R) = \{-1,0,1\}\times\mathbb{Z}_4\times\mathbb{Z}_4$ by Theorem 4.1(7). Also, $\mathrm{vnl}(R) = \mathrm{vnr}(R)\cup(1+\mathrm{vnr}(R))$ by Theorem 5.1(1), and $\mathrm{cln}(R) = \{-1,0,1,2\}\times\mathbb{Z}_4\times\mathbb{Z}_4$ by Theorem 6.1(5). Thus $\mathrm{Idem}(R) \subsetneq \mathrm{vnr}(R) \subsetneq \pi\text{-}r(R) \subsetneq \mathrm{cln}(R) \subsetneq R$ and $\mathrm{vnr}(R) \subsetneq \mathrm{vnl}(R) \subsetneq \mathrm{cln}(R)$. Note that $\pi\text{-}r(R)$ and $\mathrm{vnl}(R)$ are not comparable since $(1,2,3) \in \pi\text{-}r(R)\setminus\mathrm{vnl}(R)$ and $(2,0,1) \in \mathrm{vnl}(R)\setminus\pi\text{-}r(R)$.

We next determine the clean elements in $R[X]$, $R[[X]]$, and $R(+)M$.

Theorem 6.3. *Let R be a commutative ring.*

(1) $\mathrm{cln}(R[X]) = \{\sum a_nX^n \in R[X] \mid a_0 \in \mathrm{cln}(R),\ a_n \in \mathrm{nil}(R)$ *for every* $n \geq 1\}$.

(2) [18, Example 2] $R[X]$ *is never a clean ring.*

(3) $\mathrm{cln}(R[[X]]) = \{\sum a_nX^n \in R[[X]] \mid a_0 \in \mathrm{cln}(R)\}$.

(4) [18, Proposition 6] $R[[X]]$ *is a clean ring if and only if R is a clean ring.*

Proof. (1) We have $\mathrm{cln}(R[X]) = U(R[X]) + \mathrm{Idem}(R[X]) = \{\sum a_nX^n \in R[X] \mid a_0 = u+e$ for some $u \in U(R)$ and $e \in \mathrm{Idem}(R)$; $a_n \in \mathrm{nil}(R)$ for every $n \geq 1\} = \{\sum a_nX^n \in R[X] \mid a_0 \in \mathrm{cln}(R),\ a_n \in \mathrm{nil}(R)$ for every $n \geq 1\}$ since $U(R[X]) = \{\sum a_nX^n \in R[X] \mid a_0 \in U(R),\ a_n \in \mathrm{nil}(R)\}$ and $\mathrm{Idem}(R[X]) = \mathrm{Idem}(R)$ by Lemma 3.1.

(2) It follows directly from (1) that $X \notin \mathrm{cln}(R[X])$; so $R[X]$ is never a clean ring.

(3) It follows by definition since $U(R[[X]]) = \{\sum a_nX^n \in R[[X]] \mid a_0 \in U(R)\}$ and $\mathrm{Idem}(R[[X]]) = \mathrm{Idem}(R)$ by Lemma 3.1.

(4) This follows directly from (3). □

Theorem 6.4. *Let R be a commutative ring and M an R-module.*

(1) $\mathrm{cln}(R(+)M) = \{(r,m) \mid r \in \mathrm{cln}(R),\ m \in M\} = \mathrm{cln}(R)(+)M$.

(2) [3, Theorem 1.10] $R(+)M$ *is a clean ring if and only if R is a clean ring.*

Proof. (1) By Lemma 3.6, $\mathrm{cln}(R(+)M) = U(R(+)M) + \mathrm{Idem}(R(+)M) = \{(u,m)+(e,0) \mid u \in U(R),\ e \in \mathrm{Idem}(R),\ m \in M\} = \{(u+e,m) \mid u \in U(R),\ e \in \mathrm{Idem}(R),\ m \in M\} = \{(r,m) \mid r \in \mathrm{cln}(R),\ m \in M\} = \mathrm{cln}(R)(+)M$.

(2) This follows directly from (1). □

Does Theorem 2.5 (or Theorem 4.5) generalize to the von Neumann local elements or clean elements of R? That is, does $\{0\} \neq Z(R) \subseteq \mathrm{vnl}(R)$ (resp., $\{0\} \neq Z(R) \subseteq \mathrm{cln}(R)$) imply that R is a von Neumann local (resp., clean) ring? The next example shows that it does not.

Example 6.5. Let $A = \mathbb{Z}[[X]], M = \mathbb{Z}[[X]][1/X]/\mathbb{Z}[[X]]$ an A-module, and $R = A(+)M$. Then $\{0\} \neq \mathrm{nil}(R) = \{0\}(+)M \subsetneq Z(R) = XA(+)M \subsetneq \mathrm{cln}(R)$ by Lemma 3.6(2)(3), Theorems 6.4(1) and 6.1(3), and $\mathrm{cln}(R) = \mathrm{vnl}(R)$ by Theorem 6.1(4) and Lemma 3.6(4). However, R is not a clean ring, and thus not a von Neumann local ring by Theorem 6.4(2) since A is not a clean ring by Theorem 6.3(4).

We have $\pi\text{-}r(R) = U(R)\cup\mathrm{nil}(R)$ if and only if $\mathrm{Idem}(R) = \{0,1\}$ by Theorem 4.1(5), and $\pi\text{-}r(R) = \mathrm{vnr}(R)\cup\mathrm{nil}(R)$ if and only if either $\mathrm{Idem}(R) = \{0,1\}$ or

$\mathrm{nil}(R) = \{0\}$ by Theorem 4.4(1). We next consider to what extent these two results extend to $\mathrm{vnl}(R)$ and $\mathrm{cln}(R)$.

Theorem 6.6. *Let R be a commutative ring, and consider the following statements:*

(a) $\mathrm{vnl}(R) = U(R) \cup \mathrm{nil}(R)$.
(b) $\mathrm{cln}(R) = U(R) \cup \mathrm{nil}(R)$.
(c) $\mathrm{vnl}(R) = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$.
(d) $\mathrm{cln}(R) = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$.

Then

(1) (a)$\Leftrightarrow$(b), (c)$\Leftrightarrow$(d), *and* (a)$\Rightarrow$(c).
(2) *If any of the four statements holds, then* $\pi\text{-}r(R) = \mathrm{vnl}(R) = \mathrm{cln}(R)$.
(3) *If* (a) *or* (b) *holds, then* $\mathrm{Idem}(R) = \{0, 1\}$.
(4) *If* (c) *or* (d) *holds, then either* $\mathrm{Idem}(R) = \{0, 1\}$ *or* $\mathrm{nil}(R) = \{0\}$.

Proof. (1) Clearly (a)$\Rightarrow$(c) since $U(R) \subseteq \mathrm{vnr}(R) \subseteq \mathrm{vnl}(R)$ by Theorem 6.1(1). We show (a)$\Leftrightarrow$(b); the proof for (c)$\Leftrightarrow$(d) is similar, and hence is omitted.

(a)$\Rightarrow$(b) Suppose that $\mathrm{vnl}(R) = U(R) \cup \mathrm{nil}(R)$. Then $\mathrm{vnl}(R)$ is multiplicatively closed; so $\mathrm{cln}(R) = \mathrm{vnl}(R)$ by Theorem 6.1(8). Thus (b) holds.

(b)$\Rightarrow$(a) Suppose that $\mathrm{cln}(R) = U(R) \cup \mathrm{nil}(R)$. Since $U(R) \cup \mathrm{nil}(R) \subseteq \mathrm{vnl}(R) \subseteq \mathrm{cln}(R)$ by Theorems 5.1(3) and 6.1(1), we have $\mathrm{vnl}(R) = \mathrm{cln}(R)$. Thus (a) holds.

(2) Suppose that one of the four statements holds. Then $\mathrm{vnl}(R) = \mathrm{cln}(R)$ by (1), and $U(R) \cup \mathrm{nil}(R) \subseteq \mathrm{vnr}(R) \cup \mathrm{nil}(R) \subseteq \pi\text{-}r(R) \subseteq \mathrm{cln}(R)$ by Theorems 4.1(4) and 6.1(2). Thus $\pi\text{-}r(R) = \mathrm{vnl}(R) = \mathrm{cln}(R)$.

(3) Suppose that either (a) or (b) holds. Then $\pi\text{-}r(R) = U(R) \cup \mathrm{nil}(R)$ by (2); so $\mathrm{Idem}(R) = \{0, 1\}$ by Theorem 4.1(5).

(4) Suppose that either (c) or (d) holds. Then $\pi\text{-}r(R) = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$ by (2); so either $\mathrm{Idem}(R) = \{0, 1\}$ or $\mathrm{nil}(R) = \{0\}$ by Theorem 4.4(1). ☐

Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Then R is a Boolean ring; so $R = \mathrm{vnr}(R) = \mathrm{vnl}(R) = \mathrm{cln}(R)$. However, $U(R) \cup \mathrm{nil}(R) \subsetneq R$. Thus (c) does not imply (a) in the above theorem. Also, letting $R = \mathbb{Z}$ shows that the converses of (3) and (4) both fail.

Theorems 2.4(2) and 4.4(2) do not extend to $\mathrm{vnl}(R)$ and $\mathrm{cln}(R)$. For example, the quasilocal domain $R = \mathbb{Z}_{(2)}$ is both a von Neumann local ring and a clean ring, but it is not a total quotient ring. However, if $T(R) = R$, then certainly $R = \mathrm{vnl}(R) \cup Z(R) = \mathrm{cln}(R) \cup Z(R)$.

7 Zero-divisor Graphs

As in [10], the *zero-divisor graph* of a commutative ring R, denoted by $\Gamma(R)$, is the undirected graph with vertices $Z(R)^*$ and two distinct vertices x and y are adjacent if and only if $xy = 0$ (see [6] for a recent survey article on zero-divisor graphs). In this final section, we consider five induced subgraphs $\Gamma(\mathrm{Idem}(R))$, $\Gamma(\mathrm{vnr}(R))$, $\Gamma(\pi\text{-}r(R))$, $\Gamma(\mathrm{vnl}(R))$, and $\Gamma(\mathrm{cln}(R))$ of $\Gamma(R)$ with vertices $\mathrm{Idem}(R) \cap Z(R)^*$, $\mathrm{vnr}(R) \cap Z(R)^*$, $\pi\text{-}r(R) \cap Z(R)^*$, $\mathrm{vnl}(R) \cap Z(R)^*$, and $\mathrm{cln}(R) \cap Z(R)^*$, respectively. For $Z(R)^* \neq \emptyset$, we have $\Gamma(\mathrm{Idem}(R)) = \Gamma(R)$ (resp., $\Gamma(\mathrm{vnr}(R)) = \Gamma(R)$) if and only if R is a Boolean (resp., von Neumann regular) ring by Theorem 2.7 (resp., Theorem 2.5), and for $\mathrm{nil}(R) \subsetneq Z(R)$, we have $\Gamma(\pi\text{-}r(R)) = \Gamma(R)$ if and only if R is a π-regular ring by Theorem 4.5. Clearly $\Gamma(R)$ is the empty graph if and only if R is an integral domain. However, the above subgraphs may be empty when R is not an integral domain. For example, $\Gamma(\mathrm{Idem}(R))$ is the empty graph if and only if $\mathrm{Idem}(R) = \{0, 1\}$, and $\Gamma(\pi\text{-}r(R))$ is the empty graph when R is a quasilocal reduced ring by Theorem 2.1(5) and Corollary 4.3(3). Also, note that $\mathrm{Idem}(R) \cap Z(R)^* \neq \emptyset$ if and only if $\mathrm{vnr}(R) \cap Z(R)^* \neq \emptyset$ by Theorem 2.1(2).

We have $\Gamma(\mathrm{Idem}(R)) \subseteq \Gamma(\mathrm{vnr}(R)) \subseteq \Gamma(\pi\text{-}r(R)) \subseteq \Gamma(\mathrm{cln}(R)) \subseteq \Gamma(R)$ and $\Gamma(\mathrm{vnr}(R)) \subseteq \Gamma(\mathrm{vnl}(R)) \subseteq \Gamma(\mathrm{cln}(R))$ for any commutative ring R. Since a zero-dimensional (e.g., finite) ring is π-regular, $\Gamma(\pi\text{-}r(R)) = \Gamma(\mathrm{vnl}(R)) = \Gamma(\mathrm{cln}(R)) = \Gamma(R)$ when R is zero-dimensional. However, for $T = \mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_4$ as in Example 6.2, we have $\Gamma(\mathrm{Idem}(T)) \subsetneq \Gamma(\mathrm{vnr}(T)) \subsetneq \Gamma(\pi\text{-}r(T)) \subsetneq \Gamma(\mathrm{cln}(T)) \subsetneq \Gamma(T)$, $\Gamma(\mathrm{vnr}(T)) \subsetneq \Gamma(\mathrm{vnl}(T)) \subsetneq \Gamma(\mathrm{cln}(T))$, and $\Gamma(\pi\text{-}r(T))$ and $\Gamma(\mathrm{cln}(T))$ are not comparable. Moreover, $\Gamma(T)$ is infinite, while the other five subgraphs are all finite.

We next recall some concepts from graph theory (for any undefined notation or terminology in graph theory, see [14]). Let G be an (undirected) graph. We say that G is *connected* if there is a path between any two distinct vertices of G. For distinct vertices x and y of G, the *distance* between x and y, denoted by $d(x, y)$, is the length of a shortest path connecting x and y ($d(x, x) = 0$ and $d(x, y) = \infty$ if no such path exists). The *diameter* of G is $\mathrm{diam}(G) = \sup\{d(x, y) \mid x, y$ are vertices of $G\}$. We define the *girth* of G, denoted by $\mathrm{gr}(G)$, as the length of a shortest cycle in G, provided G contains a cycle; otherwise $\mathrm{gr}(G) = \infty$. It is well known that $\Gamma(R)$ is connected with $\mathrm{diam}(\Gamma(R)) \leq 3$ and that $\mathrm{gr}(\Gamma(R)) \leq 4$ if $\Gamma(R)$ contains a cycle (for instance, see [6, Theorems 2.2 and 2.3]). Thus $\mathrm{diam}(\Gamma(R)) \in \{0, 1, 2, 3\}$ and $\mathrm{gr}(\Gamma(R)) \in \{3, 4, \infty\}$. We next show that these two results also hold for the subgraphs $\Gamma(\mathrm{Idem}(R))$, $\Gamma(\mathrm{vnr}(R))$, and $\Gamma(\pi\text{-}r(R))$ of $\Gamma(R)$. However, the subgraphs $\Gamma(\mathrm{vnl}(R))$ and $\Gamma(\mathrm{cln}(R))$ need not be connected (see Example 7.7).

Lemma 7.1. *Let R be a commutative ring and $x \in R$.*
(1) *If $x \in \mathrm{vnr}(R) \cap Z(R)^*$, then there is $y \in \mathrm{vnr}(R) \cap Z(R)^* \backslash \{x\}$ with $xy = 0$.*
(2) *If $x \in \pi\text{-}r(R) \cap Z(R)^*$, then there is $y \in \pi\text{-}r(R) \cap Z(R)^*$ with $xy = 0$.*

Proof. (1) Let $x^2 z = x$ for $z \in R$. Then $xz \in \mathrm{Idem}(R)\backslash\{0, 1\}$, and thus $y = 1 - xz \in \mathrm{Idem}(R)\backslash\{0, 1\}$. Hence $y \in \mathrm{vnr}(R) \cap Z(R)^*$ and $xy = x(1 - xz) = x - x^2 z = 0$. We have $y \neq x$ since x is not nilpotent by Theorem 2.1(3).

(2) This is clear if $x \in \mathrm{nil}(R)$; so we may assume that x is not nilpotent. Then $x^n \in \mathrm{vnr}(R) \cap Z(R)^*$ for some $n \geq 1$ by Theorem 4.2. Thus $x^n z = 0$ for some $z \in \mathrm{vnr}(R) \cap Z(R)^*$ by (1). We may assume that $x^{n-1} z \neq 0$; hence $y = x^{n-1} z \in \pi\text{-}r(R) \cap Z(R)^*$ by Theorem 4.1(3) and $xy = x(x^{n-1} z) = x^n z = 0$. □

Theorem 7.2. *Let R be a commutative ring.*
(1) $\Gamma(\mathrm{vnr}(R))$ *is connected with* $\mathrm{diam}(\Gamma(\mathrm{vnr}(R))) \leq 3$.
(2) $\mathrm{gr}(\Gamma(\mathrm{vnr}(R))) \leq 4$ *if* $\Gamma(\mathrm{vnr}(R))$ *contains a cycle.*

Proof. (1) Let x, y be distinct elements in $\mathrm{vnr}(R) \cap Z(R)^*$ with $xy \neq 0$. By Lemma 7.1, there are $a, b \in \mathrm{vnr}(R) \cap Z(R)^*$ such that $xa = yb = 0$. If $ab \neq 0$, then $ab \in \mathrm{vnr}(R) \cap Z(R)^*$ by Theorem 2.1(2), and thus $x - ab - y$ is a path of length 2 from x to y in $\Gamma(\mathrm{vnr}(R))$. If $ab = 0$, then $x - a - b - y$ is a path of length at most 3 from x to y in $\Gamma(\mathrm{vnr}(R))$. Hence $\Gamma(\mathrm{vnr}(R))$ is connected and $\mathrm{diam}(\Gamma(\mathrm{vnr}(R))) \leq 3$.

(2) Let $a - b - c_1 - \cdots - c_n - a$ be a cycle in $\Gamma(\mathrm{vnr}(R))$. If $c_1c_n = 0$, then $a - b - c_1 - c_n - a$ is a cycle of length 4 in $\mathrm{vnr}(R)$. Suppose that $c_1c_n \neq 0$. Then $a \neq c_1c_n$ and $b \neq c_1c_n$ since $a(c_1c_n) = b(c_1c_n) = 0$ and $a, b \notin \mathrm{nil}(R)$ by Theorem 2.1(3). Since $c_1c_n \in \mathrm{vnr}(R) \cap Z(R)^*$ by Theorem 2.1(2), $a - b - c_1c_n - a$ is a cycle of length 3 in $\Gamma(\mathrm{vnr}(R))$. Thus $\mathrm{gr}(\Gamma(\mathrm{vnr}(R))) \leq 4$. □

Theorem 7.3. *Let R be a commutative ring.*

(1) $\Gamma(\mathrm{Idem}(R))$ *is connected with* $\mathrm{diam}(\Gamma(\mathrm{Idem}(R))) \leq 3$.

(2) $\mathrm{gr}(\Gamma(\mathrm{Idem}(R))) \leq 4$ *if* $\Gamma(\mathrm{Idem}(R))$ *contains a cycle.*

Proof. (1) The proof is similar to that of Theorem 7.2(1). For $x = e$ and $y = f$ with $e, f \in \mathrm{Idem}(R) \cap Z(R)^*$, let $a = 1 - e$ and $b = 1 - f$.

(2) The proof is similar to that of Theorem 7.2(2) since $\mathrm{Idem}(R) \cap Z(R)$ is multiplicatively closed and contains no nonzero nilpotent elements. □

Theorem 7.4. *Let R be a commutative ring.*

(1) $\Gamma(\pi\text{-}r(R))$ *is connected with* $\mathrm{diam}(\Gamma(\pi\text{-}r(R))) \leq 3$.

(2) $\mathrm{gr}(\Gamma(\pi\text{-}r(R))) \leq 4$ *if* $\Gamma(\pi\text{-}r(R))$ *contains a cycle.*

Proof. (1) The proof is similar to that of Theorem 7.2(1) using Lemma 7.1(2).

(2) This follows from the same proof for the girth of $\Gamma(R)$ as given in [6, Theorem 3]. Note that the simpler proof for $\Gamma(\mathrm{vnr}(R))$ given in Theorem 7.2(2) may fail for $\Gamma(\pi\text{-}r(R))$ since $\pi\text{-}r(R)$ may contain nonzero nilpotent elements. □

We next give an additional hypothesis on R which guarantees that $\Gamma(\mathrm{vnr}(R))$, and hence $\Gamma(\pi\text{-}r(R))$, $\Gamma(\mathrm{vnl}(R))$, and $\Gamma(\mathrm{cln}(R))$, all contain a 4-cycle. Note that the hypothesis that $2 \notin Z(R)$ is crucial. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_4$; then $\{0, 1\} \subsetneq \mathrm{Idem}(R)$, but $\Gamma(R)$ contains no cycles.

Theorem 7.5. *Let R be a commutative ring with $2 \notin Z(R)$ and $\{0, 1\} \subsetneq \mathrm{Idem}(R)$. Then $\Gamma(R)$ contains a 4-cycle with each vertex in $\mathrm{vnr}(R)$. In particular, $\Gamma(\mathrm{vnr}(R))$, $\Gamma(\pi\text{-}r(R))$, $\Gamma(\mathrm{vnl}(R))$, and $\Gamma(\mathrm{cln}(R))$ all have girth at most 4.*

Proof. Let $e \in \mathrm{Idem}(R) \backslash \{0, 1\}$. Then e, $1 - e \in \mathrm{vnr}(R) \cap Z(R)^*$. Since $2 \notin Z(R)$, e, $-e$, $1 - e$, and $-(1 - e) = e - 1$ are distinct elements in $\mathrm{vnr}(R) \cap Z(R)^*$. Thus $e - (1 - e) - (-e) - (e - 1) - e$ is the desired 4-cycle. The "in particular" statement is clear. □

Remark 7.6. More generally, suppose that $x, y \in Z(R)^*$ are distinct with $xy = 0$ and $y \notin \mathrm{nil}(R)$. If $2 \notin Z(R)$, then $x - y - (-x) - (-y) - x$ is a cycle of length 4 in $\Gamma(R)$. Theorem 7.5 is then the special case with $x = e$ and $y = 1 - e$ for $e \in \mathrm{Idem}(R) \backslash \{0, 1\}$.

The next example shows that $\Gamma(\mathrm{vnl}(R))$ and $\Gamma(\mathrm{cln}(R))$ need not be connected.

Example 7.7. Let $A = \mathbb{Z}[[X]]$, $I = 3XA$, and $R = A/I = \mathbb{Z}[[X]]/3X\mathbb{Z}[[X]]$. Then $\mathrm{cln}(R) = \mathrm{vnl}(R)$ by Theorem 6.1(4) since $\mathrm{Idem}(R) = \{0, 1\}$. We will show that $\Gamma(\mathrm{vnl}(R)) = \Gamma(\mathrm{cln}(R))$ is not connected. Let $a = 2X + I$, $b = 4X + I \in R$. Then $a, b \in \mathrm{vnl}(R) \cap Z(R)^*$ by Theorem 5.1(3)(8). However, there is no $c \in \mathrm{cln}(R) \cap Z(R)^*$

adjacent to either a or b since $\operatorname{ann}_R(a) = \operatorname{ann}_R(b) = 3R$. In fact, $\Gamma(R)$ is the complete bipartite graph on the two disjoint sets $C = \{Xf + I \mid f \in A,\ 3 \nmid f\} = (XR)^*$ and $D = \{3f + I \mid f \in A,\ X \nmid f\} = (3R)^*$, and $C = \operatorname{vnl}(R) \cap Z(R)^* = \operatorname{cln}(R) \cap Z(R)^*$; so $\Gamma(\operatorname{vnl}(R)) = \Gamma(\operatorname{cln}(R))$ is totally disconnected. Thus Theorem 7.2(1) does not extend to $\Gamma(\operatorname{vnl}(R))$ and $\Gamma(\operatorname{cln}(R))$.

We next give a condition which ensures that $\Gamma(\operatorname{vnl}(R))$ and $\Gamma(\operatorname{cln}(R))$ are each connected with diameter at most 3. In the next theorem, for $S \subseteq R$, we let $\Gamma(S)$ be the induced subgraph of $\Gamma(R)$ with $S \cap Z(R)^*$ its set of vertices (this notation agrees with our earlier notation for $\Gamma(\operatorname{Idem}(R)), \ldots, \Gamma(\operatorname{cln}(R))$).

Theorem 7.8. *Let R be a commutative ring and $S \subseteq R$ such that $\operatorname{nil}(R)$ is a prime ideal of R and $\operatorname{nil}(R)^* \subseteq S$. Then $\Gamma(S)$ is connected with $\operatorname{diam}(\Gamma(S)) \leq 3$, and $\operatorname{gr}(\Gamma(S)) \leq 4$ if $\Gamma(S)$ contains a cycle.*

Proof. Let $a, b \in S \cap Z(R)^*$ with $ab \neq 0$. Suppose that $a \in \operatorname{nil}(R)$. Then by the proof of [8, Lemma 2.3], there is $w \in \operatorname{nil}(R)^*$ such that $a - w - b$ is a path from a to b in $\Gamma(S)$. Now suppose that $a, b \notin \operatorname{nil}(R)$. Since $\operatorname{nil}(R)$ is a prime ideal of R and $a, b \in Z(R)^*$, there are $c, d \in \operatorname{nil}(R)^*$ such that $ac = bd = 0$. If $c = d$, then $a - c - b$ is a path from a to b in $\Gamma(S)$. If $cd \neq 0$, then $a - cd - b$ is a path from a to b in $\Gamma(S)$. Finally, if $cd = 0$ and $c \neq d$, then $a - c - d - b$ is the desired path in $\Gamma(S)$. Thus $\operatorname{diam}(\Gamma(S)) \leq 3$.

Suppose that $c_1 - c_2 - \cdots - c_n - c_1$ is a cycle in $\Gamma(S)$ with each $c_i \in \operatorname{nil}(R)^*$. Since $c_2, c_n \in \operatorname{nil}(R)^*$, by the proof of [8, Lemma 2.1], there is $w \in \operatorname{nil}(R)^*$ such that $c_2 - w - c_n$ is a path in $\Gamma(S)$. Thus $c_1 - c_2 - w - c_n - c_1$ is a cycle of length 4 in $\Gamma(S)$. Now suppose that some c_i is not nilpotent; say $c_1 \notin \operatorname{nil}(R)$. Since $\operatorname{nil}(R)$ is a prime ideal of R, we have $c_2, c_n \in \operatorname{nil}(R)^*$. Again, by the proof of [8, Lemma 2.1], there is $h \in \operatorname{nil}(R)^*$ such that $c_2 - h - c_n$ is a path in $\Gamma(S)$. But then $c_1 - c_2 - h - c_n - c_1$ is a cycle of length 4 in $\Gamma(S)$. Hence $\operatorname{gr}(\Gamma(S)) \leq 4$. □

Corollary 7.9. *Let R be a commutative ring such that $\operatorname{nil}(R)$ is a prime ideal of R.*

(1) $\Gamma(\operatorname{vnl}(R))$ *is connected with* $\operatorname{diam}(\Gamma(\operatorname{vnl}(R))) \leq 3$, *and* $\operatorname{gr}(\Gamma(\operatorname{vnl}(R))) \leq 4$ *if* $\Gamma(\operatorname{vnl}(R))$ *contains a cycle.*

(2) $\Gamma(\operatorname{cln}(R))$ *is connected with* $\operatorname{diam}(\Gamma(\operatorname{cln}(R))) \leq 3$, *and* $\operatorname{gr}(\Gamma(\operatorname{cln}(R))) \leq 4$ *if* $\Gamma(\operatorname{cln}(R))$ *contains a cycle.*

Proof. Since $\operatorname{nil}(R) \subseteq \operatorname{vnl}(R) \subseteq \operatorname{cln}(R)$ by Theorems 5.1(3) and 6.1(1), the corollary follows directly from the above theorem using $S = \operatorname{vnl}(R)$ in (1) and $S = \operatorname{cln}(R)$ in (2). □

The zero-divisor graphs of von Neumann regular rings and Boolean rings have been studied in [26], [9], [21], [22] and [23]. We next show that some of the results from [9] carry over to $\Gamma(\operatorname{Idem}(R))$ and $\Gamma(\operatorname{vnr}(R))$, but first we recall some definitions. Distinct vertices a and b of a graph G are *orthogonal*, written $a \perp b$, if a and b are adjacent and there is no vertex c of G which is adjacent to both a and b, i.e., the edge $a - b$ is not part of any triangle in G. We say that G is *complemented* if for each vertex a of G, there is a vertex b of G (called a *complement* of a) such that $a \perp b$; and

G is *uniquely complemented* if G is complemented and whenever $a \perp b$ and $a \perp c$ in G, then b and c are adjacent to exactly the same vertices in G. Then $\Gamma(R)$ is uniquely complemented if and only if either $T(R)$ is von Neumann regular or $\Gamma(R)$ is a star graph [9, Corollary 3.10]. In particular, $\Gamma(R)$ is uniquely complemented when R is von Neumann regular. We next generalize this to $\Gamma(\mathrm{Idem}(R))$ and $\Gamma(\mathrm{vnr}(R))$. This result does not extend to $\Gamma(\pi\text{-}r(R))$, $\Gamma(\mathrm{vnl}(R))$, and $\Gamma(\mathrm{cln}(R))$ since $\Gamma(\pi\text{-}r(R)) = \Gamma(R)$ for any zero-dimensional (e.g., finite) commutative ring R, and $\Gamma(\mathbb{F}_4[X]/(X^2))$ is a triangle, and thus is not complemented.

Theorem 7.10. *Let R be a commutative ring. Then $\Gamma(\mathrm{Idem}(R))$ and $\Gamma(\mathrm{vnr}(R))$ are uniquely complemented.*

Proof. We first show that $\Gamma(\mathrm{vnr}(R))$ is complemented. Let $a \in \mathrm{vnr}(R) \cap Z(R)^*$. Then $a = ue$ for some $u \in U(R)$ and $e \in \mathrm{Idem}(R)\backslash\{0,1\}$ by Theorem 2.2, and thus $b = u(1-e) \in \mathrm{vnr}(R) \cap Z(R)^*$. Then $b \neq a$, $ab = 0$, and $a + b = u \in U(R)$. Suppose that $c \in \mathrm{vnr}(R) \cap Z(R)^*$ is adjacent to both a and b in $\Gamma(\mathrm{vnr}(R))$. Then $ac = bc = 0$, hence $uc = (a+b)c = ab + ac = 0$, and thus $c = 0$, a contradiction. Hence $a \perp b$ in $\Gamma(\mathrm{vnr}(R))$ (this also follows from [9, Lemma 3.3]). We next show that $\Gamma(\mathrm{vnr}(R))$ is uniquely complemented. So suppose that $a \perp b$ and $a \perp c$ in $\Gamma(\mathrm{vnr}(R))$. It is sufficient to show that if $d \in \mathrm{vnr}(R) \cap Z(R)^*\backslash\{a,b,c\}$ with $db = 0$, then $dc = 0$. Suppose that $dc \neq 0$. Then $(dc)a = d(ca) = 0$, $(dc)b = c(db) = 0$, and $dc \in \mathrm{vnr}(R) \cap Z(R)^*\backslash\{a,b\}$ since $\mathrm{vnr}(R)$ is multiplicatively closed and $a, b \notin \mathrm{nil}(R)$. But this contradicts that $a \perp b$ in $\Gamma(\mathrm{vnr}(R))$; so $\Gamma(\mathrm{vnr}(R))$ is uniquely complemented.

The proof for $\Gamma(\mathrm{Idem}(R))$ is similar, but somewhat simpler, to that for $\Gamma(\mathrm{vnr}(R))$ with $a = e$ and $b = 1 - e$ for $e \in \mathrm{Idem}(R) \cap Z(R)^*$. □

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References

[1] E. Abu Osba, M. Henrikson, O. Alkam, Combining local and von Neumann regular rings, *Comm. Algebra* **32** (2004) 2639–2653.

[2] E. Abu Osba, M. Henrikson, O. Alkam, F.A. Smith, The maximal regular ideal of some commutative rings, *Comment. Math. Univ. Carolin.* **47** (2006) 1–10.

[3] M.S. Ahn, D.D. Anderson, Weakly clean rings and almost clean rings, *Rocky Mountain J. Math.* **30** (2006) 783–798.

[4] D.D. Anderson, V.P. Camillo, Commutative rings whose elements are a sum of a unit and an idempotent, *Comm. Algebra* **30** (2002) 3327–3336.

[5] D.D. Anderson, M. Winders, Idealization of a module, *J. Commutative Algebra* **1** (2009) 3–56.

[6] D.F. Anderson, M.C. Axtell, J.A. Stickles, Zero-divisor graphs in commutative rings, in: *Recent Developments in Commutative Algebra, Noetherian and Non-Noetherian Perspectives*, Springer-Verlag, New York, 2011, pp. 23–45.

[7] D.F. Anderson, A. Badawi, Divisibility conditions in commutative rings with zero-divisors, *Comm. Algebra* **30** (2002) 4031–4047.

[8] D.F. Anderson, A. Badawi, On the zero-divisor graph of a ring, *Comm. Algebra* **36** (2008) 3073–3092.

[9] D.F. Anderson, R. Levy, J. Shapiro, Zero-divisor graphs, von Neumann regular rings, and Boolean algebras, *J. Pure Appl. Algebra* **180** (2003) 221–241.

[10] D.F. Anderson, P.S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra* **217** (1999) 434–447.

[11] M. Contessa, On certain classes of MP rings, *Comm. Algebra* **12** (1984) 1447–1469.

[12] M. D'Anna, M. Fontana, An amalgamated duplication of a ring along an ideal: the basic properties, *J. Algebra Appl.* **6** (2007) 443–459.

[13] M. D'Anna, M. Fontana, The amalgamated duplication of a ring along a multiplicative-canonical ideal, *Ark. Mat.* **45** (2007) 241–252.

[14] R. Diestel, *Graph Theory*, Springer, New York, 1997.

[15] G. Ehrlich, Unit-regular rings, *Portugal. Math.* **27** (1968) 209–212.

[16] J.W. Fisher, R.L. Synder, Rings generated by their units, *J. Algebra* **42** (1976) 363–368.

[17] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York-Basel, 1972.

[18] J. Han, W.K. Nicholson, Extensions of clean rings, *Comm. Algebra* **29** (2001) 2589–2595.

[19] J.A. Huckaba, *Commutative Rings with Zero Divisors*, Marcel Dekker, New York-Basel, 1988.

[20] K.N. Kyun, L. Yang, On right quasi-duo rings which are π-regular, *Bull. Korean Math. Soc.* **37** (2000) 217–227.

[21] J.D. LaGrange, Complemented zero-divisor graphs and Boolean rings, *J. Algebra* **315** (2007) 600-611.

[22] J.D. LaGrange, On realizing zero-divisor graphs, *Comm. Algebra* **36** (2008) 4509–4520.

[23] J.D. LaGrange, Characterizations of three classes of zero-divisor graphs, *Canad. Math. Bull.* **55** (2012) 127–137.

[24] T.Y. Lam, *Exercises in Classical Ring Theory*, Springer, New York, 2003.

[25] J. Lambek, *Lectures on Rings and Modules*, Blaisdell Publishing Company, Waltham-Toronto-London, 1966.

[26] R. Levy, J. Shapiro, The zero-divisor graph of von Neumann regular rings, *Comm. Algebra* **30** (2002) 745–750.

[27] W.K. Nicholson, Lifting idempotents and exchange rings, *Trans. Amer. Math. Soc.* **229** (1977) 269–278.

[28] R. Raphael, Some remarks on regular and strongly regular rings, *Canad. Math. Bull.* **17** (1975) 709–712.